Danish Rational Economic Agents Model - DREAM, Version 1.2

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*This subsection reproduces the work of Schultz Møller (1993).

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Chapter 1
INTRODUCTION

This paper describes the dynamic computable general equilibrium (CGE) model of the Danish economy called DREAM (Danish Rational Economic Agents Model)[1] that is under development at Statistics Denmark[2]. The version 1.2 of the model presented here is a benchmark version; it is kept simple and all markets are competitive. Thus the present version can be considered a benchmark to which future, and more elaborate model-versions may be compared.

Compared to version 1.0 (August 12, 1997) some refinements have been made. A new utility function that has total instead of average utility as it's criterion has been chosen. This altered objective affects consumption and savings behavior quite substantially as will appear later on. Further governmental producers have been introduced in the model to be able to calibrate the model on the actual Danish data. In order to keep it simple governmental producers are assumed to perform optimization only intra- and not intertemporally. Governmental investments are assumed to be a constant share of GDP, and the governmental producers are cost minimizing within every period given the amount of physical capital. Also associated with the calibration, a destinction between taxable and non-taxable age dependent public transfers is carried through, and these are being indexed to the wage net of labor market contribution. Finally, consumption of very old persons now figures as private consumption instead of public, and transfers between the government and foreign countries are included in the model.

It should be stressed that the present version of the model and all experiments are of a preliminary nature, and therefore it is advisable to consult with the authors before

[1] The name is inspired by the (long) list of topics that the sponsors of the project would like the model to cover: "Only a dream could fulfill this".

[2] The computable general equilibrium group of Statistics Denmark was initiated by January 1st 1997. The group has gradually grown in size, and has medio May 1997 reached its equilibrium size of 5 full time employed academics and 2 part time employed research assistants.

using the material further.

1.1 Features of the model

The remaining part of this section contains a brief presentation of the features of the model; the following chapters contain a more formal presentation of the different parts of the model. The model consists of a corporate sector, households, a government and a foreign sector. The supply part of the model is rather standard and closely related to the EPRU-model (see Jensen *et al.*; 1995). The behavior of the consumers are modelled along the lines of Auerbach and Kotlikoff (1987), but is extended to allow for exogenous expected changes in the size of the household along the life-cycle, both due to the presence of children and due to expected mortality. Furthermore, the household leaves a foreseen bequest to their children.

1.1.1 Firms

The fundamental behavioral assumption of the corporate sector is that firms strive to maximize the discounted value of current and future net-of-tax-dividends to the owners of the stock of shares in the firm. It is assumed that the firm finances investments by an exogenous combination of debt and retained profits. This implies that the discounted value of the stream of dividends is positive (and thus the shares have a positive economic value). The positive discounted value of dividends is composed of the earnings on the non-debt financed part of the capital stock and the discounted value of the pure profits of the firm. Pure profits appear due to the fact, that the net production function of the firm is strictly concave, because of convex costs of installation of new capital (and also of deploying existing capital). Furthermore, these costs imply that an individual firm will want to adjust its capital stock gradually towards a stationary capital stock, if the firm is in an environment with constant prices and demand. Investments are driven by the so-called marginal q-theory of investment. Contrary to this, we abstract for the time being from any turn-over costs of labor, which implies that the employment decision does not depend directly on future prices and wages.

1.1.2 Households

The household sector consists of overlapping generations of households with a finite and deterministic time horizon. The modelling of the households is an innovation in the sense that neither the Blanchard (1985) nor the Auerbach and Kotlikoff (1987) approach to modelling overlapping generations is used. The gains from using the Blanchard approach is that it allows for analytical aggregation across generations and therefore offers computational simplicity, which significantly reduces simulation time and hardware requirements. The costs of performing analytical aggregation have been considered too high, since it limits the scope for differences between agents in the economy. Therefore the modelling is based on numerical aggregation of the behavior of the different generations.

Households are defined as representative couples with children. The size of the representative household is affected by the exogenous age and gender specific death probabilities of the members, and by the age specific fertility rates of women. The adult members of the households are divided into workers and pensioners, according to an exogenously given retirement age. The explicit incorporation of both children and a retirement period gives rise to a life-cycle motive for saving (and dissaving) in the model. At the end of the planning horizon, each generation of households leave a bequest to their children. Agents may be alive *after* the end of the planning horizon - these persons live from public transfers and take no independent economic decisions of their own, and one can think of them as living in residential homes for elderly people. Younger generations derive utility from consumption of the domestic and the foreign good, and incur disutility from time spent working. Bequest also has a positive effect on the donor's utility.

Income for the consumers arise from 8 main sources. The first is wages received for the hours worked, and the second is unemployment benefits for the hours they are unemployed. By the constrution of the model, work-sharing prevails such that everyone in the labor force is equally underemployed. The third source of income is age-dependent income transfers from the public sector, such as child-care transfers, education benefits, and sickness benefits. The fourth source of income is pension, which is given to all persons who are 61 years old or older. The pension is identical for all pensioners. The fifth source of income is a lump-sum transfer that the public sector makes in order for the budget to balance every period. The sixth type of income arises from bonds and shares held by the consumers. The seventh source of net income is transfers from abroad, such as aid transfers to developing countries and net transfers from the European Union. The final type of income arises from inheritance left by the parent household.

1.1.3 Labor market

The labor market is assumed to be competitive. Labor is a homogenous good and we abstract from individual differences in productivity. Persons, who belong to the work force and supply less labor to the market than an institutionally fixed maximum supply, are entitled to (supplementary) unemployment benefits. Thus the labor market equilibrium implies that agents voluntary reduce their individual labor supply such that underemployment and work sharing prevails. The calibrated level of unemployment in the model is thus by assumption entirely voluntary.

1.1.4 International relations

The economy is integrated in the world economy through trade and capital flows. Materials and foreign consumer goods are imported while the domestic product is exported. Domestic production is an imperfect substitute for imported goods, implying that the terms of trade is endogenous. Financial capital is assumed to be perfectly mobile internationally, and the exchange rate is fixed. These assumption, the absence of uncertainty and the presence of residence based taxation of interest income, implies that the domestic and the foreign pre-tax interest rate are equal.

The foreign demand for the domestic good can be thought of as demand functions derived from intertemporal optimization of foreigners. Assuming that foreign consumers have utility functions, which are similar to those of domestic consumers, it may be plausible to assume that the foreign demand function for the domestic good is isoelastic. For simplicity we assume that the position of the export demand curve is fixed through time.

1.1.5 The public sector

The government collects taxes, distributes income transfers and purchases the domestic good. Income transfers are assumed to be both age and gender specific. Data for these transfers in 1992 are obtained from the generational accounting project joint organized by the Danish Ministry of Finance and EPRU (see Jacobsen *et al.*; 1997).

Consistency requires that the agents do not violate their intertemporal budget constraints. For the government this implies that the discounted value of government debt has to converge to zero for time approaching infinity; violating this means, that private agents are not willing to hold government bonds. To simplify the dynamics, we assume that the government uses a lump sum transfer/tax to keep the government budget balanced *in every* period.

1.1.6 Rational expectations

Finally, as mentioned, expectations are assumed to be rational. There is no risk and uncertainty, which implies that agents have perfect foresight, except in the case of an unannounced change in a policy variable. Even after such an experiment agents are assumed to form expectations with certainty.

1.2 Simplifying assumptions in the present version of the model

Of central importance in setting up the model was keeping it simple and at the same time break the path, such that future developments face as few restrictions as possible. Of central importance to the present version of the model are the following assumptions, which may be relaxed in later versions (these assumptions are discussed in more detail in the remaining chapters):

- There are no imperfections in the markets and the entire economy is in a competitive equilibrium.

- The normalizations and the functional forms imply that there is no intertemporal speculation in the labor supply, i.e. leisure in period t is not substitutable for leisure in other periods. The size and composition of the household does not affect the labor supply, which by the specification of the utility function turns out to become identical across gender and age. There are no differences in labor productivity across individuals, which means that everyone supplies undifferentiated labor. Furthermore, there are no hiring and firing costs. All these assumptions imply that at each moment of time the labor market is modelled as a single static competitive market.

- Household behavior is simplified. Each representative household is assumed to consist of different agents according to both age and gender. Each of these types of agents may enter the utility function in different ways. How the composition of the household affects the behavior of the individuals in the household (e.g. how the number of the children in the household affects female and male labor supply) will be the subject of an empirical investigation. In the present version of the model we abstract from effects of the composition of the household on the labor supply for both male and females.

- There are two tradable goods: A domestic and a foreign

The next steps in the development of the model will challenge some of these simplifications. First, by more elaborate modelling of the labor supply, and second by introducing imperfect competition in the labor market. A third extension would be to increase the number of goods, most importantly to distinguish between non-durable goods and durable-goods (e.g. housing). Opening for imperfections in the modelling will be a gradual process, starting with the labor market.

1.3 Overview of the paper

This section gives an overview of the chapters that follows.

Chapter 2 introduces the behavior of the firms. Their behavior is not entirely as in the standard case, because there are installation costs when expanding or replacing the capital stock. This means that the profitmaximizing firms adjust their capital stock gradually, rather than at once, since gradual changes minimizes the costs of installation. When the model is not in steady state, this construct leaves room for pure profits that are distributed back to the shareholders, i.e. the consumers.

Chapter 3 presents the demographic characteristics in the model. In this respect the present model is innovative when it comes to modelling households, since the size of the household varies over the life-cycle to account for children being born into the family, children leaving home and expected mortality of the members of the household. The first part of the chapter presents the new method and compares it to the two traditional ones; Auerbach & Kotlikoff and Blanchard. Then follows a detailed description of how these synthetic households are constructed, which basically comes down to organizing men, women and children in nuclear families. Third is a discussion of how the population is predicted in the model, and thus how the important households are forecasted. Finally this chapter discusses the consequences of using the demographic structure suggested. It is also discussed why and under which circumstances the present approach is advantageous to the traditional methods.

Chapter 4 presents the problem faced by the households. Households consist of women, men and possibly children living at home. The household's problem is maximizing utility subject to a budget constraint. The overall utility function is additively separable in the utility in each moment in time. At each moment in time the consumer derives utility from consumption and incurs disutility from work. The consumers also derive utility from leaving a bequest to their children when turning 78 - this bequest is also additively separable in the utility function. It turns out that the solution to the consumer's problem is the standard Keynes-Ramsey consumption smoothing rule.

Chapter 5 describes the aggregate variables in the model. It is also discussed how the model is closed, which here amounts to describing the behavior of the foreign sector and the public sector and specifying the equilibrium conditions for the goods and the labor market.

Chapter 6 contains an analytical description of the stationary state of the model. It is demonstrated that the stationary state is unique, partly analytically and partly numerically. First it is shown mathematically that in stationary state there exists an aggregate version of the model. Then it is shown numerically that with the current calibration of the model, i.e. with the current choice of parameters, there is only one equilibrium.

Chapter 7 discusses how the model is calibrated. First it is discussed how calibrating a dynamic CGE-model differs from calibrating a static model. Here it is done by calibrating the model to a base year (1995) but adjusting the data set according to what a hypothetical stationary state for 1995 would have looked like. Unlike in static models, it is not possible to calibrate the model exactly to the data set of a specific year; this may violate the intertemporal consistency requirements in the model, because the data set is inconsistent with the assumption of the stationary state.

Chapter 8 reports the results of 12 simulations carried out with the model. The experiments shed some light on how different parts of the model react to changes in policy variables. The first 8 of the experiments focus on the impacts of changes in various tax rates influencing the behavior of households and firms in different ways. The next two of the experiments consist of a reduction in the average pensioner age and a reduction of fertility. Finally, the last two experiments illustrate the effects of an increase in the public expenditure and of an increase in the age dependent transfers.

It should be stressed that although empirical estimates are used, the present version of the model cannot be regarded as an empirical model of the Danish economy. The calibrated model should instead be thought of as a parametrized theoretical model and the simulations as a way of exploring the properties of the model in a setup where the proportions of the model resemble those of the Danish economy.

The optimal introduction to the material is not necessarily obtained by starting with chapter 2 and reading the chapters in order, since some chapters are more difficult than others. On the other hand some chapters need to be understood before others. Chapters 2 and 5 can be read alone, and chapter 3 is a prerequisite for chapter 4. Only having understood these chapters will it be possible to understand chapter 6, where everything comes together in a description of the steady state.

For the novice that needs an introduction to the model and the issues that can be investigated using it, it is recommended to take a look at chapter 8, before proceeding. This chapter contains a large number of experiments accompanied by verbal and graphical explanations. This chapter gives a good idea of what is going on, and should work as an appetizer to read the main model chapters (chapters 2 to 6). Note however, that the experiments presented in this chapter are conducted on an earlier version of the model - namely version 1.0.

1.4 An important note about the dating convention

Throughout the following dating convention is used: Variables are dated according to end of period convention. This means that stock variables that are active in period e.g. $t+1$, are nominated t, since stocks are updated at the end of period. Flow variables of period t are of course nominated t. As an exception to the dating rule, we assume that people are born in the beginning of the period. Finally, a policy shock which hits the economy in period t, is assumed to take place in the beginning of the period, i.e. before decisions about the current flow variables are taken. This dating convention is illustrated in figure 1

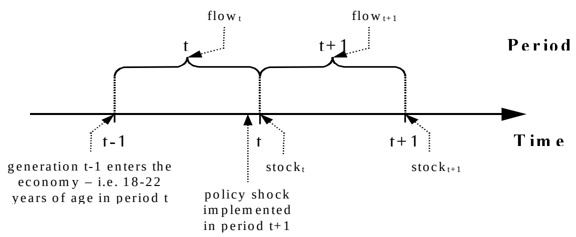

Figure 1 Illustration of the end of period dating rule used in the paper

In the model 1995 is considered period 0, and thus time 0 is ultimo 1995. This means that the first unknown or endogenously determined variables are flows from period 1 and stocks at time 1.

Chapter 2
PRODUCERS

This chapter describes the behavior of the producers. We distinguish between two categories of producers: private producers (also labelled firms) and governmental producers. Their behaviour is specified in much the same way. One very important exception is that firms are assumed to perform intertemporal optimization while governmental producers are not.

The main reason for distinguishing between private and governmental producers in this version, which is otherwise highly aggregated with respect to producers, is to obtain a more accurate calibration of the assets of households. The dominant item of these are the households' claim on the value of the private capital stock. By distinguishing between private and governmental producers, the considerable value of the governmental capital stock can be excluded from the assets of households.

Inside each category of producers we assume that the agents are identical. Since all firms in the economy are assumed to be identical, and since each firm is a price taker in both input and output markets, it is sufficient to analyze the behavior of a representative firm. The first section outlines the firm's problem and the second section goes through the first order conditions characterizing the solution to the problem. The third section outlines the governmental producer's much simpler problem and then examines the solution.

2.1 The firm's problem

The value of firms is determined from an arbitrage condition, which states that the gains from investing in shares must pay the same yield after tax as investing in bonds.

The assumption of perfect capital mobility and absence of uncertainty implies that domestic and foreign bonds are perfect substitutes. Therefore a tax-adjusted version of the uncovered interest parity (UIP) holds in the model. Absence of exchange rate movements implies that the domestic interest rate after tax in equilibrium is equal to the foreign interest rate after tax. With a residence based taxation of personal capital income this implies that the domestic (pre-tax) interest rate is equal to the foreign (pre-tax) interest rate.

Since Danish personal capital income taxation is a residence based tax, we assume as a starting point that the marginal Danish investor is a domestic citizen, who is subject to Danish domestic tax laws.[1] This assumption is not innocent for two reasons. First, foreign investors may be the marginal source of funds, if domestic tax laws give higher incentives to investment in bonds than does foreign tax laws (assuming all countries use a residence based tax). In this case foreign tax rates are the relevant ones. Second, a large part of Danish savings take place in pension funds, which are not subject to the same tax laws as citizens. The tax laws give pension funds incentive to invest in shares.

The consequence of the assumption made may be that the effect of changes in personal income taxation on the market value of shares may be exaggerated by the model.

Given the assumption that the marginal investor is subject to Danish tax laws and the fact that there is no risk in the model, the marginal investor will be indifferent between investing in bonds and in shares if

$$r_s \left(1 - t_s^r\right) V_{s-1} = \left(1 - t_s^d\right) D_s + \left(1 - t_s^g\right) \left(V_s - V_{s-1}\right) \tag{2.1}$$

where r_s is the interest rate equal to the world interest rate, which in a world with perfect mobility of financial capital is exogenously given for the small open economy. V_s is the (end of period) value of the firm. D_s is the dividends. The tax rates are: t_s^r, tax rate on interest income, t_s^d, tax rate on dividend income, and, t_s^g, tax rate on capital gains.

The left hand side is the opportunity cost of holding shares, whereas the right hand side is the sum of dividends and capital gains after tax, which is equal to the total income from holding the value V_{s-1} in shares.

Observe that due to the possibility of different tax rates the investor is not indifferent between a unit increase in the (pre-tax) dividends and a unit increase in the (pre-tax) market value of shares. A unit decrease in the dividends may be compensated by $\frac{1-t_s^d}{1-t_s^g}$ units increase in the market value of shares. Similarly, a 1 percentage point increase

[1] By personal capital income taxation we mean: taxation of personal interest income, dividend taxation and taxation of capital gains. Observe that corporate income taxation is source based.

in the (pre-tax) interest rate requires an increase in the market value of shares of $\frac{1-t_s^r}{1-t_s^g}$ percentage points for the arbitrage condition (2.1) to hold. These tax adjustment factors will appear frequently in the expressions concerning the behavior of the firm.

Rearranging the difference equation (2.1) and leading the resulting equation one period yields

$$V_s = \frac{1}{\left(1 + r_{s+1}\frac{\left(1-t_{s+1}^r\right)}{\left(1-t_{s+1}^g\right)}\right)} \left[\frac{\left(1-t_{s+1}^d\right)}{\left(1-t_{s+1}^g\right)} D_{s+1} + V_{s+1}\right] \tag{2.2}$$

Leading the expression for V_s one period and inserting the resulting value of V_{s+1} back into the expression above yields

$$V_s = \frac{1}{\left(1+r_{s+1}\frac{\left(1-t_{s+1}^r\right)}{\left(1-t_{s+1}^g\right)}\right)} \left[\frac{\left(1-t_{s+1}^d\right)}{\left(1-t_{s+1}^g\right)} D_{s+1} + \left(\frac{1}{\left(1+r_{s+2}\frac{\left(1-t_{s+2}^r\right)}{\left(1-t_{s+2}^g\right)}\right)} \left[\frac{\left(1-t_{s+2}^d\right)}{\left(1-t_{s+2}^g\right)} D_{s+2} + V_{s+2}\right]\right)\right]$$

Solving forward yields

$$V_s = \sum_{t=s+1}^{\infty} \frac{1-t_t^d}{1-t_t^g} D_t \prod_{v=s+1}^{t} \frac{1}{1+r_v\left(\frac{1-t_v^r}{1-t_v^g}\right)} \tag{2.3}$$

The market value of the firm is given as a tax adjusted discounted stream of dividends. Observe that given our dating conventions, V_s is the end-of-period market value of the firm. We assume that each firm strives to maximize the beginning-of-period value of their outstanding stock of shares, which is given as the end-of-period market value of share plus the tax-adjusted dividends in the current period. Thus we assume that the firm maximizes the tax adjusted stream of current and discounted future dividends from the firm. The managers of the firms maximize this value given the assumption of perfect foresight and subject to the production technology. To solve this problem we define dividends at time t, D_t, as

$$\begin{aligned} D_t &= \left(1-t_t^c\right)\left(p_t^P Y_t^P - p_t^{PM} M_t^P - \left(1+t_t^a\right) W_t L_t^P - r_t B_{t-1}^c\right) \\ &\quad - p_t^{PI} I_t^P + t_t^c \hat{\delta}_t \hat{K}_{t-1} + \left(B_t^c - B_{t-1}^c\right) \end{aligned} \tag{2.4}$$

where p_t^P is the (producer)price of firm's output, Y_t^P is the (net) production of the firm, p_t^{PM} is the price index of materials used by the firm, M_t^P is the input of materials, t_t^a is the payroll tax rate, W_t is the wage rate, L_t^P is firm employment, B_{t-1}^c is the stock of corporate debt at the beginning of period t, p_t^{PI} is the price index of the firm's investments, I_t^P is the firm's investment, $\hat{K}_{t-1}$ is the book value of the capital stock given the tax system, $\hat{\delta}_t$ is the rate of depreciation allowed by the tax system and t_t^c is the corporate tax rate[2].

To avoid the corner solution properties of the optimal financing decision of the firms (which appears due to the non neutrality of the tax system and the absence of uncertainty in the model), we model corporate debt as a fixed fraction of the replacement value of the capital stock

$$B_t^c = gp_t^{PI} K_t^P, \quad 0 < g < 1 \tag{2.5}$$

where K_t^P is the capital stock of the firm.[3]

Inserting (2.5) into (2.4) implies that investments are financed by retentions plus debt, and the residual cash flow is always distributed to the owners. This is the so-called "new view of dividend taxation".

Observe, that the exogenous financing rule of the firm implies that dividends will temporarily become negative in case of a sufficiently large boost in investment. This is due to the fact that the financing rule fixes the debt-financed part of the expansion in investment to g. In this case the firm collects negative dividends from its shareholders.[4]

The evolution of the capital stock and the book value of the capital stock given the tax system, are assumed to be subject to exponential decay and are thus given by[5]

$$K_t^i = \left(1 - \delta^i\right) K_{t-1}^i + I_t^i, \quad 0 < \delta^i < 1, \quad i = P, G \tag{2.6}$$

$$\hat{K}_t = \left(1 - \hat{\delta}_t\right) \hat{K}_{t-1} + p_t^{PI} I_t^P, \quad 0 < \hat{\delta}_t < 1 \tag{2.7}$$

[2] Corporate taxation is non-negative and there is a limited possibility of deduction of past deficits according to Danish tax laws. In the stationary state the relation implies that the depreciation allowance is equal to the direct cost of investment, $\hat{\delta}\hat{K} = p^I \left(\delta K\right)$. See equation (2.6) and (2.7). This assures a non-negative tax revenue in the stationary state.

The fact that the relation (2.4) may yield negative tax revenues in case of sharp recessions, thus implicitly implies that the firms in the model are allowed unlimited carry-forward of tax losses with full imputation of interest.

[3] Observe, that the definition of the stock of corporate debt deviates from the standard dating procedure of this paper. This implies that it is not the "true" replacement value.

[4] In the experiments performed in the present paper this phenomenon does not appear.

[5] The superscript i indicates whether the variable represents the private producer ($i = P$) or the governmental producer ($i = G$). Some variables as for example $\widehat{K}_t$ are only defined for the private sector and for convenience they are not supplied with such a superscript.

Figure 2 illustrates the assumed technology of the producers. It is assumed that the technology of the firm and the governmental producer can be represented by the same functional form. The upper part of figure 2 outlines the assumed production function which is specified as two-factor CES (sub) production functions nested as indicated at the figure. At the top level materials is combined with value added to produce gross output. Materials are obtained by combining governmentally produced materials and privately produced materials. Finally, privately produced materials can be either of domestic or foreign origin.

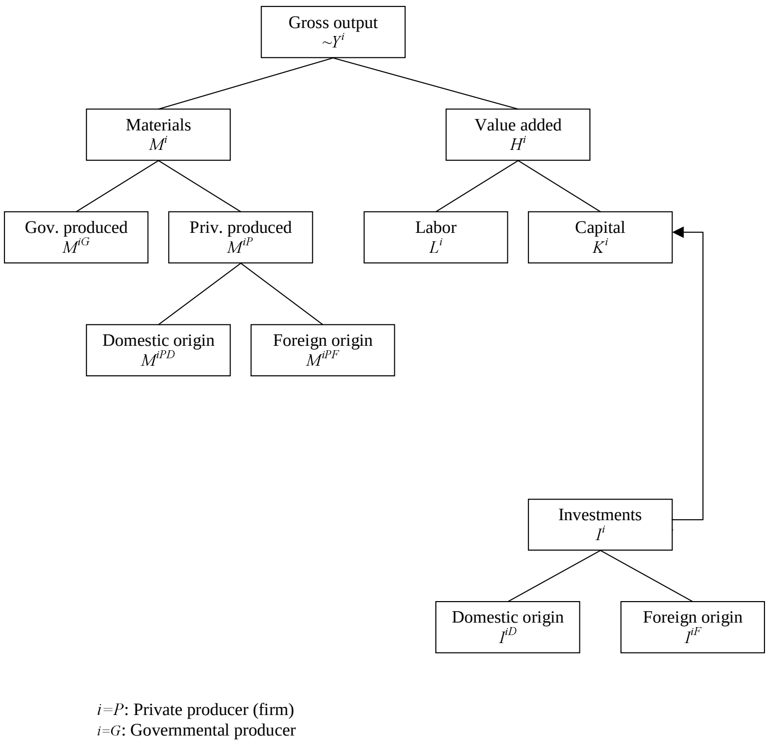

Figure 2 The technology of the producers

There are no imported materials which are classified as being governmentally produced. Value added is an aggregate of labor and capital. The capital stock is changed by investments, and figure 2 also indicates that the aggregate index of firms' investments is a mix of investment goods of domestic and foreign origin. There are no investment goods produced by the government sector (either domestic or foreign). Stated formally the net production, Y_t^i, is given as

$$Y_t^i = F^i\left(M_t^i, K_{t-1}^i, L_t^i\right) - \Phi^i(I_t^i, K_{t-1}^i), \quad i = P, G \tag{2.8}$$

where $\Phi^i(I_t^i, K_{t-1}^i)$ is the cost of installation of capital, which is specified below.[6] The gross production function $\tilde{Y}_t^i = F^i\left(M_t^i, H_t^i\right)$ exhibits constant returns to scale and is given as a nested CES function which may be written as

$$F^i\left(M_t^i, H_t^i\right) = \left[\mu_{iYM}\left(M_t^i\right)^{\frac{\sigma_{iY}-1}{\sigma_{iY}}} + \mu_{iYH}\left(H_t^i\right)^{\frac{\sigma_{iY}-1}{\sigma_{iY}}}\right]^{\frac{\sigma_{iY}}{\sigma_{iY}-1}}, \quad i = P, G \tag{2.9}$$

with

$$H_t^i = \left[\mu_{iYHK}\left(K_{t-1}^i\right)^{\frac{\sigma_{iYH}-1}{\sigma_{iYH}}} + \mu_{iYHL}\left(L_t^i\right)^{\frac{\sigma_{iYH}-1}{\sigma_{iYH}}}\right]^{\frac{\sigma_{iYH}}{\sigma_{iYH}-1}}, \quad i = P, G \tag{2.10}$$

$$M_t^i = \left[\mu_{iYMP}\left(M_t^{iP}\right)^{\frac{\sigma_{iYM}-1}{\sigma_{iYM}}} + \mu_{iYMG}\left(M_t^{iG}\right)^{\frac{\sigma_{iYM}-1}{\sigma_{iYM}}}\right]^{\frac{\sigma_{iYM}}{\sigma_{iYM}-1}}, \quad i = P, G \tag{2.11}$$

$$M_t^{iP} = \left[\mu_{iYMPD}\left(M_t^{iPD}\right)^{\frac{\sigma_{iYMP}-1}{\sigma_{iYMP}}} + \mu_{iYMPF}\left(M_t^{iPF}\right)^{\frac{\sigma_{iYMP}-1}{\sigma_{iYMP}}}\right]^{\frac{\sigma_{iYMP}}{\sigma_{iYMP}-1}}, \quad i = P, G \tag{2.12}$$

where we define H_t^i as a CES index of labor and capital used in production in sector i ($i = P$ indicates the private sector and $i = G$ indicates the governmental sector), M_t^i is a CES index of privately and governmentally produced materials used in sector i, while M_t^{iP} is a CES index of privately produced materials of domestic, M^{iPD}, and foreign, M^{iPF}, origin used in sector i, and M_t^{iG} is governmentally produced materials used in sector i which are all of domestic origin. In the CES functions $\mu_j > 0$ for $j \in \{iYM,\ iYH,\ iYHK,\ iYHL,\ iYMP,\ iYMG,\ iYMPD,\ iYMPF\}$ are weight parameters. $\sigma_{iY} > 0$ (and $\neq 1$) is the elasticity of substitution between materials and the capital-labor aggregate. Similarly $\sigma_{iYH} > 0$ (and $\neq 1$) is the elasticity of substitution between capital and labor, $\sigma_{iYM} > 0$ (and $\neq 1$) is the elasticity of substitution between privately and governmentally produced materials, while $\sigma_{iYMP} > 0$ (and $\neq 1$) is the elasticity of substitution between privately produced materials of domestic origin and foreign origin. If the elasticity of substitution between any of the two pairs approaches unity, the associated CES function reduces to a Cobb-Douglas function.

The function representing the cost of installation of capital in the private sector, $\Phi^P\left(I_t^P, K_{t-1}^P\right)$, is strictly convex in I_t^P and homogenous of degree 1. Φ^P has the following functional form

$$\Phi^P(I_t^P, K_{t-1}^P) = \phi^P\left(\frac{|I_t^P|}{K_{t-1}^P}\right)^{\iota}|I_t^P|, \ \iota > 0 \tag{2.13}$$

[6] Installation costs are assumed to be zero for the governmental producer.

The function indicates that the costs of installation not only depends on the amount of investment, but also on the size of the existing capital stock, in the way that costs diminish the bigger K_{t-1}^P is. The numeric signs around the investments make sure that the costs are positive both in case of positive and negative gross investments.

The index of the investment good used in sector i is defined as a CES aggregate of the domestic and the foreign good

$$I_t^i = I^i\left(I_t^{iD}, I_t^{iF}\right) = \left[\mu_{iID}\left(I_t^{iD}\right)^{\frac{\sigma_{iI}-1}{\sigma_{iI}}} + \mu_{iIF}\left(I_t^{iF}\right)^{\frac{\sigma_{iI}-1}{\sigma_{iI}}}\right]^{\frac{\sigma_{iI}}{\sigma_{iI}-1}}, \quad i{=}P,G$$

where I_t^{iD}, I_t^{iF} are the parts of investment which are produced domestically respectively abroad. $\mu_{iID} > 0$, $\mu_{iIF} > 0$, are weight parameters, and $\sigma_{iI} > 0$ (and $\neq 1$) is the elasticity of substitution between the domestic and the foreign part of the investment index.

Given that the representative firm as mentioned is a price taker in all markets, we may state the maximization problem of the firm at time s as a two stage problem, where the firm in the first stage chooses the optimal values of the indexes, I_t^P and M_t^P and the input of labor, L_t^P. At the second stage the firm chooses the optimal compositions of the indexes, I_t^P and M_t^P. The problem at the first stage may be written as follows

$$\begin{aligned}
&\max_{\left(L_t^P, I_t^P, M_t^P\right)_{t=s}^{\infty}} \quad \frac{1-t_s^d}{1-t_s^g} D_s + \sum_{t=s+1}^{\infty} \frac{1-t_t^d}{1-t_t^g} D_t \prod_{v=s+1}^{t} \frac{1}{1+r_v\left(\frac{1-t_v^r}{1-t_v^g}\right)} \\
&s.t. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.14) \\
D_t &= (1-t_t^c)\left(p_t^P Y_t^P - p_t^{PM} M_t^P - (1+t_t^a) W_t L_t^P - r_t B_{t-1}^c\right) \\
&\quad - p_t^{PI} I_t^P + t_t^c \hat{\delta}_t \hat{K}_{t-1} + B_t^c - B_{t-1}^c \\
Y_t^P &= F^P\left(M_t^P, K_{t-1}^P, L_t^P\right) - \Phi^P(I_t^P, K_{t-1}^P) \\
K_t^P &= \left(1-\delta^P\right) K_{t-1}^P + I_t^P \\
\hat{K}_t &= \left(1-\hat{\delta}_t\right)\hat{K}_{t-1} + p_t^{PI} I_t^P \\
B_t^c &= g p_t^{PI} K_t^P
\end{aligned}$$

Now insert (2.6) into (2.5) and further insert this rewritten equation for the stock of corporate debt together with (2.8) and (2.5) lagged one period into (2.4). The Hamiltonian associated with the problem above is then given by

$$\mathcal{H}\left(K_{t-1}^P, \hat{K}_{t-1}^P, L_t^P, I_t^P, M_t^P, \lambda_{1t}, \lambda_{2t}\right) = \tag{2.15}$$
$$R_t \tfrac{1-t_t^d}{1-t_t^g}\left\{(1-t_t^c)\left(p_t^P\left[F^P\left(M_t^P, K_{t-1}^P, L_t^P\right) - \Phi^P(I_t^P, K_{t-1}^P)\right] - p_t^{PM} M_t^P - (1+t_t^a) W_t L_t^P\right.\right.$$
$$\left.\left. - r_t g p_{t-1}^{PI} K_{t-1}^P\right) - p_t^{PI} I_t^P + t_t^c \hat{\delta}_t \hat{K}_{t-1} + g p_t^{PI}\left[I_t^P + \left(1-\delta^P\right) K_{t-1}^P\right] - g p_{t-1}^{PI} K_{t-1}^P\right\}$$
$$+ R_t \lambda_{1t}\left[I_t^P + \left(1-\delta^P\right) K_{t-1}^P\right] + R_t \lambda_{2t}\left[p_t^{PI} I_t^P + \left(1-\hat{\delta}_t\right)\hat{K}_{t-1}\right]$$

where L_t^P, I_t^P and M_t^P are the control variables that we wish to maximize for, K_{t-1}^P and $\hat{K}_{t-1}$ are the state variables and λ_{1t} and λ_{2t} are the shadow prices on the state variables in the order just mentioned. The compound interest factor R_t is defined as follows

$$R_t = \prod_{v=s+1}^{t} \frac{1}{1 + r_v\left(\frac{1-t_v^r}{1-t_v^g}\right)} \quad for \quad t \geq s+1, \quad R_t = 1 \quad for \quad t = s \tag{2.16}$$

2.2 First order conditions to the firm's problem

Applying Pontryagin's Maximum Principle we find that the first order conditions are given by[7]

$$\frac{\partial F^P}{\partial L^P}\left(M_t^P, K_{t-1}^P, L_t^P\right) = (1+t_t^a)\frac{W_t}{p_t^P} \tag{2.17}$$

$$\frac{\partial F^P}{\partial M^P}\left(M_t^P, K_{t-1}^P, L_t^P\right) = \frac{p_t^{PM}}{p_t^P} \tag{2.18}$$

$$\frac{1-t_t^d}{1-t_t^g}\left(1 - g + (1-t_t^c)\frac{p_t}{p_t^{PI}}\frac{\partial \Phi^P}{\partial I^P}(I_t^P, K_{t-1}^P)\right) = \frac{\lambda_{1t}}{p_t^{PI}} + \lambda_{2t} \tag{2.19}$$

$$\frac{1-t_t^d}{1-t_t^g}\left\{(1-t_t^c)\, p_t^P\left[\frac{\partial F^P}{\partial K^P}\left(M_t^P, K_{t-1}^P, L_t^P\right) - \frac{\partial \Phi^P}{\partial K^P}(I_t^P, K_{t-1}^P) - r_t g \frac{p_{t-1}^{PI}}{p_t^P}\right]\right.$$
$$\left. + g p_t^{PI}\left(1-\delta^P\right) - g p_{t-1}^{PI}\right\} = r_t \frac{1-t_t^r}{1-t_t^g}\lambda_{1(t-1)} + \delta^P \lambda_{1t} - \left(\lambda_{1t} - \lambda_{1(t-1)}\right) \tag{2.20}$$

$$\frac{1-t_t^d}{1-t_t^g} t_t^c \hat{\delta}_t = r_t \frac{1-t_t^r}{1-t_t^g}\lambda_{2(t-1)} + \hat{\delta}_t \lambda_{2t} - \left(\lambda_{2t} - \lambda_{2(t-1)}\right) \tag{2.21}$$

[7] For a presentation of the propositions concerning necessary conditions in discrete dynamic optimization problems see e.g. Berck and Sydsæter (1993).

The first derivatives of the installation costs function are given by

$$\frac{\partial \Phi^P}{\partial I^P}(I_t^P, K_{t-1}^P) = (1+\iota)\frac{\Phi^P(I_t^P, K_{t-1}^P)}{I_t^P} \tag{2.22}$$

$$\frac{\partial \Phi^P}{\partial K^P}(I_t^P, K_{t-1}^P) = -\iota\frac{\Phi^P(I_t^P, K_{t-1}^P)}{K_{t-1}^P} \tag{2.23}$$

The first two conditions are standard static conditions. Equation (2.17) states that the marginal product of labor in optimum is equal to the real marginal cost of labor. Observe that employment taxes (payroll taxes) shift the demand for labor inward/downward in a standard labor demand diagram. The nested CES specification of the production function implies that labor demand is given implicitly as a function of the net production and the index of domestic input.

Similarly, equation (2.18) implicitly gives the demand for materials. The equation states, that in optimum the marginal product of materials is equal to the real price of materials. The equation governing the optimal investment decision is given in (2.19). The interpretation of this equation is also that in optimum the marginal cost of investment is equal to the marginal benefits from investment. The marginal cost is given on the left-hand side. The costs are the sum of the direct costs of investment to the owners of the firm, and the indirect costs associated with the loss of production, due to the installation of the additional capital equipment. Both types of costs are corrected by the tax adjustment factor, which applies to the stream of dividends. Observe that increasing the debt ratio of the firm reduces the direct cost to the owners, as they have to finance a smaller fraction of the investment. The benefits of an investment are twofold. First, it adds to the capital stock of the firm. The shadow price, λ_{1t}, is the marginal value of an additional unit of capital to the owners of the firm. Second, it adds to the depreciation allowance of the firm. The value of the depreciation allowance increases with the direct cost of the additional capital unit. The shadow price, λ_{2t}, measures the marginal value of a unit depreciation allowance to the owners of firm. The total benefit is the sum of these two effects. The determination of the expressions for the two shadow prices are given in the two final first order conditions. These are the conditions for the stock variables of the problem[8].

First, the shadow price of depreciation allowance to the owners of the firm, λ_{2t} is found as the solution to the difference equation (2.21), which is independent of other

[8] The expressions are deduced from the necessary conditions $\frac{\partial \mathcal{H}_t}{\partial K_{t-1}} = R_{t-1}\lambda_{1(t-1)}$ and $\frac{\partial \mathcal{H}_t}{\partial \hat{K}_{t-1}} = R_{t-1}\lambda_{2(t-1)}$ respectively.

endogenous variables. To calculate the expression we solve equation (2.21) for $\lambda_{2(t-1)}$ and lead the expression by one period. This yields

$$\lambda_{2t} = \frac{1}{\left(1 + \frac{1-t^r_{t+1}}{1-t^g_{t+1}} r_{t+1}\right)} \left[\frac{1 - t^d_{t+1}}{1 - t^g_{t+1}} t^c_{t+1} \hat{\delta}_{t+1} + \left(1 - \hat{\delta}_{t+1}\right) \lambda_{2t+1}\right] \tag{2.24}$$

Solving this difference equation forward yields

$$\lambda_{2t} = \frac{1}{1 - \hat{\delta}_t} \sum_{s=t+1}^{\infty} \frac{1 - t^d_s}{1 - t^g_s} t^c_s \hat{\delta}_s \prod_{v=t+1}^{s} \frac{\left(1 - \hat{\delta}_{v-1}\right)}{\left(1 + \frac{1-t^r_v}{1-t^g_v} r_v\right)} \tag{2.25}$$

Thus λ_{2t} is given as the tax-adjusted discounted stream of the value of depreciation allowance of a unit of capital in the entire time horizon. The discounting contains both the depreciation of the book value and tax adjusted rate of interest. Thus λ_{2t} measures the increase in the market value of the firm which is due to the depreciation allowances of a marginal unit of capital that is kept in the firm forever.

Similarly the expression (2.20) may be solved for $\lambda_{1(t-1)}$. Leading this expression yields

$$\lambda_{1t} = \frac{1}{\left(1 + \frac{1-t^r_{t+1}}{1-t^g_{t+1}} r_{t+1}\right)} \left(X_{t+1} + \left(1 - \delta^P\right) \lambda_{1(t+1)}\right) \tag{2.26}$$

where

$$\begin{aligned} X_t \;\; = \;\; & \frac{1 - t^d_t}{1 - t^g_t} \left(1 - t^c_t\right) p^P_t \left[\frac{\partial F^P}{\partial K^P} \left(M^P_t, K^P_{t-1}, L^P_t\right) - \frac{\partial \Phi^P}{\partial K^P} \left(I^P_t, K^P_{t-1}\right)\right] - \\ & \frac{1 - t^d_t}{1 - t^g_t} g p^{PI}_{t-1} \left(\left(1 - t^c_t\right) r_t - \left(\frac{p^{PI}_t}{p^{PI}_{t-1}} \left(1 - \delta^P\right) - 1\right)\right) \end{aligned}$$

Solving this difference equation yields

$$\lambda_{1t} = \frac{1}{1 - \delta^P} \sum_{s=t+1}^{\infty} X_s \prod_{v=t+1}^{s} \frac{1 - \delta^P}{\left(1 + \frac{1-t^r_v}{1-t^g_v} r_v\right)} \tag{2.27}$$

λ_{1t} is given as the discounted stream of the variable X_s, where the discounting contains both physical depreciation and the tax-adjusted rate of interest. X_s is the tax-adjusted marginal effect on the current dividends of an increase in the stock of capital. A marginal capital unit first of all affects the dividends, through the marginal revenue product of capital. The marginal revenue product has two components: A standard direct effect on gross production and an indirect effect through a reduction in the cost of installation of new capital units. Secondly, through the exogenous financial rule, an increase in the capital stock increases the stock of debt and therefore the interest payments of the firm, which reduces the tax adjusted dividends. Finally, as the firm debt-finances a fixed ratio of the nominal value of capital stock, an increase in the capital stock, increases the capital gain (or loss) from holding capital, which generates increased dividends through increased debt of the firm.

Thus λ_{1t} measures the increase in the market value of the firm which is due to the current and discounted future increase in the tax-adjusted value of net production corrected for increased financial costs.

Observe the difference between (2.25) and (2.27): The shadow price of the capital stock, λ_{1t} is affected by the exogenous debt ratio of the firm, whereas the shadow price of the depreciation allowance, λ_{2t} is not. The latter is due to the fact, that the depreciation allowance increases the earnings after corporate taxation of the firm, irrespectively of how the capital stock is financed. Thus depreciation allowance is capitalized into increased value of the stock of outstanding shares.

The first order conditions (2.20) and (2.21) may also be interpreted directly as user cost expressions (or arbitrage conditions). Equation (2.20) states, that in optimum the marginal increase in the dividend from a marginal capital unit (the left hand side of the equation) is equal to the user cost of holding capital (the right hand side). The user cost of holding capital (the right hand side) consists of the sum of the tax adjusted foregone interest payments and the (true) costs of physical depreciation plus a possible capital loss due to the change in the shadow price of capital to the owners. Equation (2.21) has a similar interpretation.

2.2.1 The demand system of the firm

The demand for the inputs of the firm can be decribed by a standard CES-demand system. To do this we introduce a new variable MPK_t^i, which is the marginal product of capital in the gross production function

$$\frac{\partial F^i}{\partial K^i}\left(M_t^i, K_{t-1}^i, L_t^i\right) \equiv MPK_t^i \ , \qquad i = P, G \tag{2.28}$$

Substituting into (2.20), we get the following equation

$$\begin{aligned}
&\frac{1-t_t^d}{1-t_t^g}\Big\{(1-t_t^c)\left[p_t^P MPK_t^P - p_t^P \frac{\partial \Phi^P}{\partial K^P}(I_t^P, K_{t-1}^P) - r_t g p_{t-1}^{PI}\right] \\
&+ g p_t^{PI}\left(1-\delta^P\right) - g p_{t-1}^{PI}\Big\} \\
= \;& r_t \frac{1-t_t^r}{1-t_t^g}\lambda_{1(t-1)} + \delta^P \lambda_{1t} - \left(\lambda_{1t} - \lambda_{1(t-1)}\right) \qquad (2.29)
\end{aligned}$$

From (2.17) and (2.18) we have that[9]

$$p_t^i \frac{\partial F^i}{\partial L^i}\left(M_t^i, K_{t-1}^i, L_t^i\right) = (1+t_t^a) W_t \ , \qquad i = P, G \qquad (2.30)$$

$$p_t^i \frac{\partial F^i}{\partial M^i}\left(M_t^i, K_{t-1}^i, L_t^i\right) = p_t^{iM} \ , \qquad i = P, G \qquad (2.31)$$

(2.28), (2.30) and (2.31) shows that the optimal factor demands can be regarded as the solutions to the following profit maximization problem

$$\max_{K_{t-1}^i, L_t^i, M_t^i} \quad p_t^i \widetilde{Y}_t^i - \left(p_t^i MPK_t^i K_{t-1}^i + (1+t_t^a) W_t L_t^i + p_t^{iM} M_t^i\right) \ , \qquad i = P, G \quad (2.32)$$

where the expression $p_t^i MPK_t^i$ can be regarded as a price of capital. (2.28), (2.30) and (2.31) are the standard necessary conditions for a solution to (2.32) given the definition of MPK_t^i. Therefore theorem 3 of appendix B, giving the CES standard factor demand functions, can be applied. Taking the nest structure of the gross production function into acount, we obtain the factor demands

$$\begin{aligned}
K_{t-1}^i &= (\mu_{iYHK})^{\sigma_{iYH}} \left(\frac{p_t^i MPK_t^i}{p_t^{iH}}\right)^{-\sigma_{iYH}} H_t^i \ , \qquad i = P, G \qquad (2.33) \\
L_t^i &= (\mu_{iYHL})^{\sigma_{iYH}} \left(\frac{(1+t_t^a) W_t}{p_t^{iH}}\right)^{-\sigma_{iYH}} H_t^i \ , \qquad i = P, G \qquad (2.34) \\
p_t^{iH} &= \left[(\mu_{iYHK})^{\sigma_{iYH}} \left(p_t^i MPK_t^i\right)^{1-\sigma_{iYH}} + (\mu_{iYHL})^{\sigma_{iYH}} \left[(1+t_t^a) W_t\right]^{1-\sigma_{iYH}}\right]^{\frac{1}{1-\sigma_{iYH}}} , \\
i &= P, G \qquad (2.35)
\end{aligned}$$

and

$$\begin{aligned}
M_t^i &= (\mu_{iYM})^{\sigma_{iY}} \left(\frac{p^{iM}}{p_t^i}\right)^{-\sigma_{iY}} \tilde{Y}_t^i \ , \qquad i = P, G \qquad (2.36) \\
H_t^i &= (\mu_{iYH})^{\sigma_{iY}} \left(\frac{p_t^{iH}}{p_t^i}\right)^{-\sigma_{iY}} \tilde{Y}_t^i \ , \qquad i = P, G \qquad (2.37) \\
p_t^i &= \left[(\mu_{iYM})^{\sigma_{iY}} \left(p_t^{iM}\right)^{1-\sigma_{iY}} + (\mu_{iYH})^{\sigma_{iY}} \left(p_t^{iH}\right)^{1-\sigma_{iY}}\right]^{\frac{1}{1-\sigma_{iY}}} , \quad i = P, G \qquad (2.38)
\end{aligned}$$

[9]Formally, (2.17) and (2.18) has not been derived for $i = G$. However, they are standard optimality requirements which are easily shown to apply to the governmental producer, given the assumptions of instantantanous adjustment in the demand for labor and materials (as opposed to capital).

The interpretation is straightforward. Consider for example the top nest (2.36)-(2.38). For a given[10] gross production, $\widetilde{Y}_t^i$, it determines the demand for the index of materials, M_t^i, and the value added index, H_t^i. The position of the demand curve (2.36) for the index of materials, M_t^i, depends on the level of gross output, $\widetilde{Y}_t^i$, and the weight parameter, μ_{iYM}, raised to the power σ_{iY}. The slope of the demand curve depends on minus the elasticity of substitution, $-\sigma_{iY}$. Note that since the materials price index, p_t^{iM}, also effects the gross output index, p_t^i, cf. (2.38), the price elasticity differs from $-\sigma_{iY}$ and also depends on the weight parameters. The demand curve (2.36) moves as p_t^{iM} changes. The demand curve (2.37) for the value added index, H_t^i, is interpreted similarly. (2.38) determines the price index for gross output, p_t^i, by equating it to the unit costs leaving no excess profits.[11]

Given the demand for the value added index, H_t^i, determined in this way, (2.33)-(2.35) determine the demand for the inputs labor, L_t^i, and capital K_t^i. In the short run, where the capital stock is given, the marginal product of capital, MPK_t^i, must in fact assume that value which assures that the given stock of capital, K_{t-1}^i, is actually demanded. This feeds back on the price of value added, p_t^{iH}, cf. (2.35), and the output price, p_t^i, cf. (2.38). For firms, the marginal product of capital, MPK_t^P, affects their shadow price of marginal capital, λ_{1t}, through (2.29). This has further repurcussions for firms' investments, I_t^P, cf. (2.19), which feeds back on their capital stock.

2.2.2 Intratemporal optimization

Given that the optimal level of M_t^i has been determined, the demand for the privately respectively the publicly produced part of the index can be found by minimizing the cost of obtaining the index. The demands are the solutions to the minimization problem

$$\begin{aligned}&\min_{M_t^{iP},\,M_t^{iG}} \quad \left(p_t^{iMP} M_t^{iP} + p_t^{iMG} M_t^{iG}\right) \ , \qquad i = P, G \\ &s.t. \\ &M_t^i = \left[\mu_{iYMP}\left(M_t^{iP}\right)^{\frac{\sigma_{iYM}-1}{\sigma_{iYM}}} + \mu_{iYMG}\left(M_t^{iG}\right)^{\frac{\sigma_{iYM}-1}{\sigma_{iYM}}}\right]^{\frac{\sigma_{iYM}}{\sigma_{iYM}-1}} \ , \qquad i = P, G\end{aligned} \tag{2.39}$$

[10] The gross production is of course not given exogenously to the firms but affected by firm behavior. In chapter 5 the equilibrium condition equating supply (output) to demand is displayed.

[11] In the terminology of standard static CGE-models, (2.38) is a zero profit condition - given that the price of capital is equal to $p_t^i MPK_t^i$. Alternatively one could say that $p_t^i MPK_t^i$ has been defined such that (2.38) acts as a zero profit condition.

where M_t^i is considered given, M_t^{iP} is the index of materials used in sector i but produced in the private sector, p_t^{iMP} is the associated price index, M_t^{iG} is the index of materials used in sector i but produced in the public sector, and p_t^{iMG} is the price of publicly produced materials used in sector i. Again using Appendix B, theorem 1, we have that the demand for privately produced materials is equal to

$$M_t^{iP} = (\mu_{iYMP})^{\sigma_{iM}} \left(\frac{p_t^{iMP}}{p_t^{iM}}\right)^{-\sigma_{iYM}} M_t^i \ , \qquad i = P, G \tag{2.40}$$

The corresponding condition for the publicly produced materials becomes

$$M_t^{iG} = (\mu_{iYMG})^{\sigma_{iM}} \left(\frac{p_t^{iMG}}{p_t^{iM}}\right)^{-\sigma_{iYM}} M_t^i \ , \qquad i = P, G \tag{2.41}$$

and the price index of materials is determined as

$$\begin{aligned} p_t^{iM} &= \left[(\mu_{iYMP})^{\sigma_{iYM}} \left(p_t^{iMP}\right)^{1-\sigma_{iYM}} + (\mu_{iYMG})^{\sigma_{iYM}} \left(p_t^{MG}\right)^{1-\sigma_{iYM}}\right]^{\frac{1}{1-\sigma_{iYM}}} , \\ i &= P, G \end{aligned} \tag{2.42}$$

Given that the optimal levels M_t^{iP} have been determined, the demand for the domestic respectively the foreign part of the index may be found by minimizing the cost of obtaining the specific index. The demands are the solutions to the minimization problems

$$\begin{aligned} &\min_{M_t^{iPD},\, M_t^{iPF}} \quad \left(1 + t_t^{iM}\right)\left(p_t^P M_t^{iPD} + M_t^{iPF}\right) \ , \qquad i = P, G \\ &s.t. \\ &M_t^{iP} = \left[\mu_{iYMPD}\left(M_t^{iPD}\right)^{\frac{\sigma_{iYMP}-1}{\sigma_{iYMP}}} + \mu_{iYMPF}\left(M_t^{iPF}\right)^{\frac{\sigma_{iYMP}-1}{\sigma_{iYMP}}}\right]^{\frac{\sigma_{iYMP}}{\sigma_{iYMP}-1}} , \ i = P, G \end{aligned} \tag{2.43}$$

where M_t^{iP} is considered given and t_t^{iM} is the tax rate on material goods used in sector i. The domestic and the foreign material are both simple non-composite goods each having the same price in all applications. The price of the foreign good is implicitly assumed to be equal to one. Again using appendix B we have that the demand for domestically, privately produced materials becomes

$$M_t^{iPD} = (\mu_{iYMPD})^{\sigma_{iYMP}} \left(\frac{\left(1 + t_t^{iM}\right) p_t^P}{p_t^{iMP}}\right)^{-\sigma_{iYMP}} M_t^{iP} \ , \qquad i = P, G \tag{2.44}$$

The corresponding condition for materials produced privately abroad is

$$M_t^{iPF} = (\mu_{iYMPF})^{\sigma_{iYMP}} \left(\frac{\left(1 + t_t^{iM}\right)}{p_t^{iMP}}\right)^{-\sigma_{iYMP}} M_t^{iP} \ , \qquad i = P, G \tag{2.45}$$

and the price index of privately produced materials is determined as

$$\begin{aligned} p_t^{iMP} &= \left(1{+}t_t^{iM}\right)\left[\left(\mu_{iYMPD}\right)^{\sigma_{iYMP}}\left(p_t^P\right)^{1-\sigma_{iYMP}} + \left(\mu_{iYMPF}\right)^{\sigma_{iYMP}}\right]^{\frac{1}{1-\sigma_{iYMP}}}, \quad (2.46) \\ i &= P, G \end{aligned}$$

The price index for the governmentally produced materials used in sector i is simply determined by inflating the output price of the governmental producer, p_t^G, with the tax rates on materials inputs in sector i

$$p_t^{iMG} = \left(1 + t_t^{iM}\right) p_t^G \ , \qquad i = P, G$$

Finally, the demand for the domestic respectively the foreign part of the index of investments, I_t^i, may analogously be found by minimizing the cost of obtaining the specific index. The demands are the solutions to the minimization problems

$$\begin{aligned} &\min_{I_t^{iD}, I_t^{iF}} \quad \left(1 + t_t^{iI}\right)\left(p_t^P I_t^{iD} + I_t^{iF}\right) \ , \qquad i = P, G \\ &s.t. \qquad (2.47) \\ &I_t^i = \left(\mu_{iID}\left(I_t^{iD}\right)^{\frac{\sigma_{iI}-1}{\sigma_{iI}}} + \mu_{iIF}\left(I_t^{iF}\right)^{\frac{\sigma_{iI}-1}{\sigma_{iI}}}\right)^{\frac{\sigma_{iI}}{\sigma_{iI}-1}} \ , \qquad i = P, G \end{aligned}$$

where I_t^i is considered given, and t_t^{iI} is the tax rate on investment goods. The solutions are

$$I_t^{iD} = \left(\mu_{iID}\right)^{\sigma_{iI}}\left(\frac{\left(1 + t_t^{iI}\right) p_t^P}{p_t^{iI}}\right)^{-\sigma_{iI}} I_t^i \ , \qquad i = P, G \qquad (2.48)$$

$$I_t^{iF} = \left(\mu_{iIF}\right)^{\sigma_{iI}}\left(\frac{\left(1 + t_t^{iI}\right)}{p_t^{iI}}\right)^{-\sigma_{iI}} I_t^i \ , \qquad i = P, G \qquad (2.49)$$

$$p_t^{iI} = \left(1 + t_t^{iI}\right)\left(\left(\mu_{iID}\right)^{\sigma_{iI}}\left(p_t^P\right)^{1-\sigma_{iI}} + \left(\mu_{iIF}\right)^{\sigma_{iI}}\right)^{\frac{1}{1-\sigma_{iI}}} \ , \qquad i = P, G \qquad (2.50)$$

2.2.3 Govermental producers

Following the national accounts conventions, the governmental sector in DREAM produces mainly non-marketed goods and services of which an overwhelming part are delivered directly to public consumption. Publicly owned firms which mainly produces for a market are classified as part of the private sector.

As described in the preceeding section, the governmental production and factor demands are modelled in much the same way as the production and factor demands of the private sector firms. One important exception is that we do not assume any intertemporal optimization of the public sector. Public (gross) investments are instead assumed to amount to a constant share of overall GDP. Together with the capital accumulation identity this determines the development of the public capital stock. Thus, the value of the marginal product of capital can deviate between the private and the public sector because the public capital stock is not determined by (intertemporal) optimization. This is a crude way to model the behaviour of policy-makers, which just assures that the size of the public sector relative to the private sector is broadly unchanged.[12] For simplicity it is assumed that there are no installation costs of investments in the public sector.

Given the politically determined investments and capital stock, the public demands for the remaining inputs labor and materials of different origins are determined along the lines of the private sector firms, i.e. it is assumed that the public sector minimizes costs subject to a nested CES-technology and given the capital stock. Thus even though the public sector does not substitute between capital and labor plus materials in an economic optimal way it is still assumed that it substitutes between labor and various materials in an optimal way - given the capital stock.

Formally, the expenditures on public investments are determined as a constant share of overall nominal GDP

$$p_t^{GI} I_t^G = \kappa \cdot \widehat{Y}_t, \quad 0 \leq \kappa < 1 \tag{2.51}$$

where I_t^G is the index of public investments, p_t^{IG} is the associated price index, $\widehat{Y}_t$ is nominal GDP as defined in (5.21) in chapter 5, and κ is the share. The public capital stock, K_t^G, accumulates according to (2.6).

For convenience, it assumed that there are no installation costs for the public sector. This implies that gross production and net production are identical, i.e. $\tilde{Y}_t^G = Y_t^G$.

The factor demand equations of the governmental sector are specified analogously to

[12] Analogously, public consumption is assumed to amount to a constant fraction of GDP. All items of the public budget are determined without any reference to intertemporal optimization, cf. chapter 5.3.

the factor demand equations of the private sector, cf. the preceding section 2.2. Note that the resulting CES factor demand equation for public capital, cf. (2.33)

$$K_{t-1}^G = (\mu_{GYHK})^{\sigma_{GyH}} \left(\frac{p_t^G MPK_t^G}{p_t^{GH}} \right)^{-\sigma y_{GH}} H_t^G \qquad (2.52)$$

determines the marginal product of public capital, MPK_t^G, even in the long run, because the capital stock, K_t^G, is determined from (2.6).

Chapter 3
DEMOGRAPHIC STRUCTURE

Modelling overlapping generations of households in computable general equilibrium models is typically based on either Blanchard (1985) or Auerbach and Kotlikoff (1987). The present model, however, distinguishes itself by developing a new approach to modelling households. In the following this new approach is presented briefly, and an overview of the two traditional methods is given.

In the Blanchard model, the household at each moment of time faces a (constant) probability of death and each household is engaged in a deal with a life insurance company. The insurance company takes over the stock of wealth as the household dies. To compensate for this, the household receives an annuity premium, as long as it stays alive.[1] By assumption the deal is actuarial fair, such that there are no profits (and no administration costs) in the insurance sector. The construction implies that, there is no bequest in these models. One major advantage of modelling household behavior this way is that analytical aggregation over households of different ages is possible. Unfortunately this is only possible if some rather restricting assumptions are made on the differences between agents in the economy.[2]

Auerbach and Kotlikoff assume in their model, that households of different ages differ by the level of labor productivity, (assumed to be a hump-shaped function of the age of the generation) and that each household has a deterministic and finite time horizon. Bequest is absent in their original model, such that each generation has a stock of wealth of zero, when they die. Later extensions like Steigum and Steffensen (1990), Kenc and Perraudin (1997) and Lassila, Palm and Valkonoen (1997) allow

[1] The Danish Labour Market Pensions Funds can be seen as institutions which resembles the insurance companies in the Blanchard model. There are two differences: First, the annuity premium is accumulated until the member reaches the retirement age. Second, the insurance in the Danish pension funds is only partial, as the insurance deal only concerns the part of the household's stock of wealth, which is the accumulated stream of savings in the pension fund.

[2] The behavior of consumers in the EPRU-model is based on this idea. See Jensen *et al.* (1996).

for a positive bequest motive. Finally, Chauveau and Louffir (1997) allow both for a positive bequest motive and for an uncertain lifetime.

The inclusion of bequest in the model has the effect that the households' incentive to save is increased. Thus at any given moment of time the stock of financial wealth in the economy is increased *ceteris paribus*. Furthermore, inclusion of bequest implies that the effects of unanticipated shocks to the economy become more gradual, because the generations alive react by changing the bequest as well as the consumption in each period. Since the bequest is the inheritance to the future generation, the total income of this generation is affected directly. Thus the bequest behavior of the second generation influences the third generation and so on. In this way, the higher weight each generation places on bequest in the utility function, the longer will the economy's reaction to a shock be.

In the initial version of the present model, labor productivity is assumed to be identical across generations. However, the present paper distinguishes itself by considering the household as a decision-unit consisting of a couple of adults that may have children living at home. The representative household of each generation as a decision unit has a deterministic finite planning horizon. The individual members of the household face an uncertain lifetime although the household as such does not. This implies that the surviving part of the household retains undivided possession of the stock of wealth, until the horizon of the household expires.

The construction of these households is described in detail in the following section, and the rest of the chapter is organized as follows: The second section describes how the size and composition of the households are forecasted in the present model. In the third section a description of the constructed data series is given, and finally, the last section discusses the implications of the demographic structure used in the present model. It is also discussed why and under which circumstances the present approach is advantageous to the traditional methods.

3.1 Construction of the representative households

The starting point is the statistics on population by gender and age in 1995 for Denmark. This is used to divide the population into generations according to birth-year. One generation consists of all persons born in a specific year. To convert the generation of individuals into (synthetic) representative households we use the following assumptions and generalizations:

1. There is one representative household in each generation.
2. A representative household is defined as couples of adult males and adult females, with children below the age of 18 years belonging to the household of their mother.
3. The household size is measured in adult-equivalents, as will be explained later.
4. Since the two parties need not be born in the same year, the age of the household is defined in terms of the age of the female.
5. The planning horizon for a household (a woman) is 60 years. Correspondingly the expected lifetime for a 18 year old woman was 78 years in 1995.
6. The household (as a decision unit) expects with certainty to survive until the end of the planning horizon. However, the *size* of the representative household is expected to be reduced over time according to the individual probability of death of the members of the household. This is equivalent to an assumption that a surviving member of the household will always exist and that this member retains undivided possession of the estate. The fact, that children are born into the household and later leave home at the age of 18 (to form a household of their own), also affects the size of the parent household.
7. The adults in the household younger than 61 are active in the labor market, and subsequently they retire and become pensioners.
8. Persons surviving the planning horizon of the household (i.e. when the woman turns 78)[3] are assumed only to recieve social security pensions, pay taxes and consume the rest, i.e. the do not supply labor, save or possess assets.

Some of these ideas are illustrated in figure 3 below. The course of life is split into 2 main phases: childhood (below 18 years) and adulthood (18 years and above). Children belong to their parent household, which is done by associating them with their mother. Adulthood can be divided into two phases: one of activity and one of

[3] Observe, that men who are older than 77 years but married to women, who are younger than 78 years, do not stop economic activities before their wifes turn 78 years.

passivity. Only the active group is interesting as independent economic agents, since the passive group is taking no economic decisions of their own. The government takes care of them, and one can think of them as living in residential homes for elderly people.

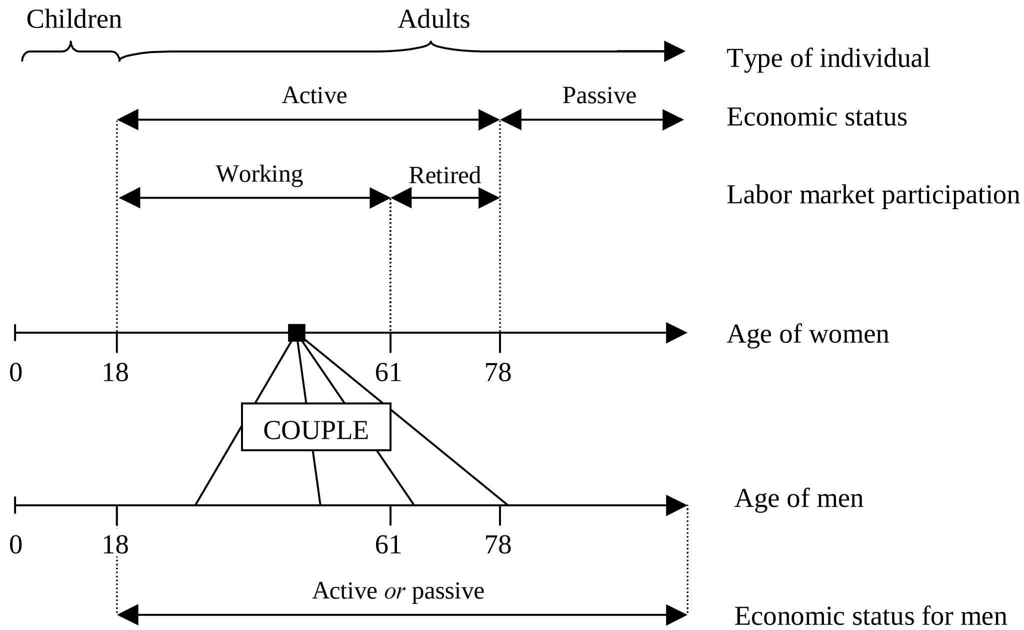

Figure 3 The course of life for women and men in the model

Furthermore, it is possible to divide the active period of life in two phases: the first with activity in the labor market (i.e. working or being unemployed) and the second period as retired and not supplying labor. As will be accounted for more thoroughly in the next chapter, people are assumed to retire at age 61, which means that the period with activity in the labor market lasts from 18 to 61 years.

Later it will be explained how households, as previously mentioned, are formed from individuals, but for now it will suffice to think of a group of women at a specific age (e.g. all 47 years old) being joined with a group of men. The point illustrated in figure 3 is that even though the group of women is born in the same year, this is not the case for the group of men to whom they are joined. For the group of women indicated by the black square, this means that since *they* are economically active, their partners are also economically active, even if these (men) are more than 77 years old. However, even though the women are participating in the labor market (as indicated in the black square in figure 3), this need not be the case for the group of men belonging to their household; some can be working, some can be retired and some can even be more than 78 years old.

This illustrates the somewhat asymmetric treatment of men and women when it comes to determining whether they are active or passive. When women turn 78 they become passive and stop all economic activity, and so does their partner irrespectively of whether or not they have turned 78 themselves. Thus whether a man older than 77 is active or passive is *not* determined by his own age, but rather by the age of the woman he is assigned to.[4]

However, symmetry between the genders prevails when it comes to age and labor market participation. Whether a man is working or retired from the labor market, is related to his *own* age, and not the age of his household. Note, that this does not preclude the possibility, that the elasticity of his labor supply with respect to wages are influenced by the age of the woman to which he is assigned.

To sum up, it is important to bear in mind that it is the age of the woman that determines the household age, and this age determines whether the household is active or passive; for this purpose the age of the man does not matter. Thus, since each household is defined in terms of the age of the woman, it is necessary to assign men and children to a woman. This is done in a manner described in the next sub section.

Before we proceed note that the population variables of the model will be introduced when they will be needed as we go along. The notation is rather complex - several subscripts and superscripts will be used separately and in combination. Therefore an explanation for the construction of this notation and an overview of the variables are given in appendix A.

3.1.1 Men, women and the couple-matrix

Assigning men to a household means matching men and women. In other words we define a mapping, such that all men born within a one-year interval, are distributed to the female population. Thus every woman is not married to one particular man, but rather to a distribution of men. From Statistics Denmark (SD) we obtain a matrix measuring the number of couples distributed over the combinations of ages of the two parties in the couple. In the statistical material the definition of a couple,

[4] Note, that technically the construction implies that it is possible for an "economically passive" man to turn "economically active" by becoming married to a woman who is younger than 78 years.

is simply a man and a woman sharing the same address. Here we confine ourself to combinations of men and women belonging to the age groups $A = \left[18, \tilde{A}\right]$ and $B = [18, 77]$ respectively. The observations are given by the following matrix

$$\Omega_t = \begin{bmatrix} N^{couple}_{18,18,t} & N^{couple}_{18,19,t} & \cdots & N^{couple}_{18,77,t} \\ N^{couple}_{19,18,t} & N^{couple}_{19,19,t} & \cdots & N^{couple}_{19,77,t} \\ \vdots & \vdots & \ddots & \vdots \\ N^{couple}_{\tilde{A},18,t} & N^{couple}_{\tilde{A},19,t} & \cdots & N^{couple}_{\tilde{A},77,t} \end{bmatrix} \tag{3.1}$$

where $N^{couple}_{a,b,t}$ is the number of couples where the man is a years old and the woman is b years old. Note that the last column of the matrix, is the expected lifetime for a woman in the model, after which her household ceases to exist.

The matrix is constructed using data from SD[5]. The matrix Ω_{1995} can be determined for the part of the population, where men and women form couples. Approximately 64 percent of the population in the age groups A and B for men and women respectively had formed couples in 1995.

The remaining part of the population in these age groups is either single, or live in a type of relationship other than the one defined above. Since every person has to be attached to a household we need to specify a way to distribute these last 36 percent of the relevant population group.

3.1.2 Normalizing the couple-matrix

A simple way to proceed is first to normalize the matrix, Ω_t, defined above, such that the rows sum to unity. More formally divide every element in the matrix by the sum of all the elements in the row in question (i.e. by the number of men in the given generation, who have formed a couple). The resulting normalized couple matrix, $\bar{\Omega}_t$,

[5]The raw data from 1995 are used, except for women below 25 years. The distribution of their male partners age, is different from the rest of the population in two ways; their partners are significantly older, and the percentage of women, who are in a couple is low for these age groups. Therefore the pattern of relative difference in age for 25-year-old women, are assumed to apply to all women between 18 and 24 years of age. Observations for women above 77 years are dismissed, since the households' planning horizon is 60 years. Consistency with generalizations mentioned previously, requires that couples where the man or the woman (or both) are below 18 years of age are dismissed as well.

is given by

$$\bar{\Omega}_t = \begin{bmatrix} \omega_{18,18,t} & \omega_{18,19,t} & \cdots & \omega_{18,77,t} \\ \omega_{19,18,t} & \omega_{19,19,t} & \cdots & \omega_{19,77,t} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{\tilde{A},18,t} & \omega_{\tilde{A},19,t} & \cdots & \omega_{\tilde{A},77,t} \end{bmatrix} \tag{3.2}$$

where $\omega_{a,b,t} \in R_{[0,1]}$ and $\sum_{b \in B} \omega_{a,b,t} = 1$ for all a, t

An element $\omega_{a,b,t}$ in the matrix will then define the share of men a years of age (and part of a couple) who is married to a woman of age b at time t. Thereby the matrix can row-wise be interpreted as a discrete probability function (PDF) describing the distribution of women that are assigned to a vintage of men.[6]

What we do next is simply to take all men in the age group A, and distribute each of the generations to the different ages of women according to the corresponding PDF from the matrix $\bar{\Omega}_t$, as if they were all part of a couple. However, since the number of men and women in the population are not identical, this implies that more or less than one man can be assigned to each woman (on average).

We assume from here that the matrix $\bar{\Omega}_t$ is stationary through time, such that we have $\bar{\Omega} \equiv \bar{\Omega}_t = \bar{\Omega}_{1995}$ for all t. Let $\vec{N}_t^M$ be a row vector (of dimension $\tilde{A} - 17$), where each element equals the number of men in the age group given by the column (the first column is men at 18 years), at time t. An element of this vector is called $N_{a,t}^M$, where a is the age of the man in a given year t. Thus the sum of the elements, equals the total number of men between 18 and $\tilde{A}$ years in a given year. Further let $\vec{N}_t^F$ denote the row vector (of dimension $77 - 17$), consisting of the elements, $N_{b,t}^F$ ($b =$ 18, 19, ..., 77), which indicate the number of females aged b at time t.

Assigning the men to households (defined by the woman's age) is done simply by matrix multiplication. Thus $\vec{N}_t^M \times \bar{\Omega} = \vec{N}_t^{MF}$ gives the number of men, distributed after the household age (instead of their own). An element in the vector $\vec{N}_t^{MF}$ is called $N_{b,t}^{MF}$ and measures the number of men, who at time t is matched with a female of age b. Thus adding women, measured by their own age, and men defined by the women's age reduces to adding the two vectors $\vec{N}_t^{MF}$ and $\vec{N}_t^F$ (both vectors belong to R^{60}). An element in this vector sum is called $N_{b,t}^{AF}$ and measures the number of survived adults at time t in the representative household of age b (i.e. the generation born at time $t - b$).

[6] If the elements in the diagonal were unity, i.e. $\omega_{a,a,t} = 1$ for all a, this would imply that only men and women born in the same year, could make couples.

The number of persons who belong to the sum of these synthetic households is not exactly identical to the number of adult men and adult women in the real population. The difference is due to truncations at the two ends of the distribution of men. Most importantly not all young men (mostly of age 18, 19 and 20 years) are assigned to women, who are more than 17 year old, as the average man is approximately 3 years older than the average women in the household. Since women of 17 years and younger by assumption do not form households of their own, this implies that men assigned to women of these age groups are also excluded from the data set. At the other end of the distribution a similar but quantitatively much less outspoken problem arises. In this case some men younger than 78 year are married to women who are 78 year or more. These men are considered "economically passive" as the age of the household is defined by the age of the woman.

Apart from the mentioned truncations, the main effect of assigning men to women is that the age distribution of the population is now "standardized" and measured in terms of the age of the woman. The fact that women are married to a distribution of men, rather than a particular man is illustrated in figure 4.

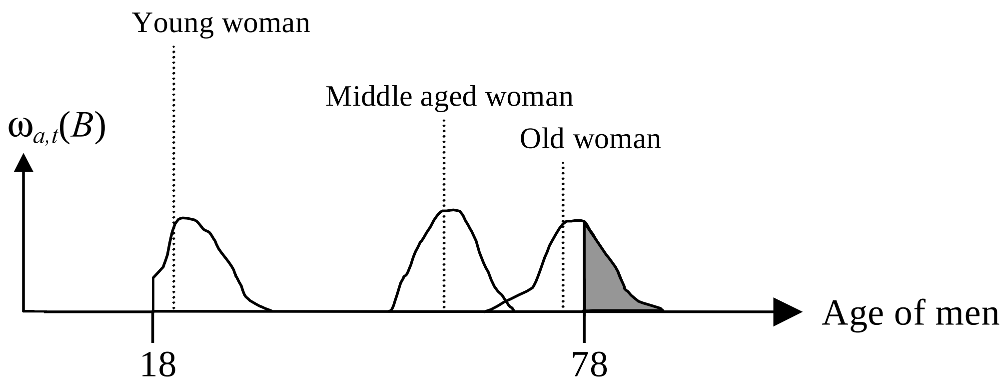

Figure 4 The distributions of men that 3 age groups of women are matched with

The figure shows an example of this PDF for a young, a middle-aged and an old woman (however younger than 78), where the density function is denoted $\omega_{a,t}(B)$. The PDF is unique for each generation of women, and need not be normal, symmetric or even single-peaked[7]. Note again that even though an adult woman is between 18 and 77, this need not be the case for all the men in her partner-PDF. In the PDF to the right, this is shown by the gray area; it contains men that are older than 77, but are married to women that are not retired, and thus these men are still economic

[7] Note that the PDF is in fact discrete, even though it is depicted as continous.

active (consuming, saving etc. but not supplying labor). In the other end of the scale this is not the case, as illustrated by the PDF to the left. None of the men associated with the young women can be less than 18 years old and therefore the distribution has been cut off to the left of 18[8].

In principle a very old woman can be married to a very young man and vice versa, but this is a rare occurance. In the figure this is illustrated by a frequency of zero[9] for some age intervals for men associated with a woman of a given age.

3.1.3 Children

Assigning children to a household means matching child and mother - this is done in the following manner: Let $\vec{N}_t^{\tilde{C}}$ be a row vector with 18 elements, containing the number of children grouped by age (first column is children between 0 and 1 years, and the last contains the number of children between 17 and 18 years). Let $\vec{N}_t^{C}$ be a corresponding row vector consisting of the same children converted into a measure of so-called adult-equivalent units using the function $\vec{N}_t^{C} = A^{eq}\left(\vec{N}_t^{\tilde{C}}\right)$.[10] The reason why this transformation is made is to mirror the fact that children are not requiring the same level of consumption as adults.

Let $\psi_{c,t}(B)$ denote the discrete probability function of the mother's age[11] for children c years old at time t, $c = 0, 1, ..., 17$, and let 49 years be the maximal maternal age when giving birth. Note that 49 years is defined as the maximum in the publication of the statistics. These maternal vectors of densities can be written as a maternal matrix, Ψ_t, consisting of 18 rows and 60 columns. The 18 rows reflect that we want to assign 18 generations of children, where each row represents a PDF[12]. The 60 columns reflect the 60 generations of women in the ages between 18 and 77, where only the women at 66 years or younger are potential mothers for these children. The Ψ_t-matrix

[8] This point is discussed further in Appendix D.

[9] Strictly speaking, it is *the limit* of the frequency (the "probability") that is zero.

[10] In this first version of the model, the function is reduced to $\vec{N}_t^{C} = 0.5 \cdot \vec{N}_t^{\tilde{C}}$.

[11] Or the age of the household, the child is part of, in the case where the mother dies before the child is grown up.

[12] Note that some men are still living at home even though they are between 18 and 28. The reason for this is explained in Appendix D.

looks like this

$$\Psi_t = \begin{bmatrix} \psi_{0,18,t} & \psi_{0,19,t} & \cdots & \psi_{0,35,t} & \cdots & \psi_{0,49,t} & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \psi_{1,19,t} & \cdots & \psi_{1,35,t} & \cdots & \psi_{1,49,t} & \psi_{1,50,t} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & \psi_{17,35,t} & \cdots & \psi_{17,49,t} & \psi_{17,50,t} & \cdots & \psi_{17,66,t} & 0 & \cdots \end{bmatrix} \tag{3.3}$$

where $\sum_{b \in B} \psi_{c,b,t} = 1$ for all c, t.

For children born in the base year (1995) the distribution of the mothers' age is known with certainty from Statistics Denmark. This distribution is shown in figure 5.

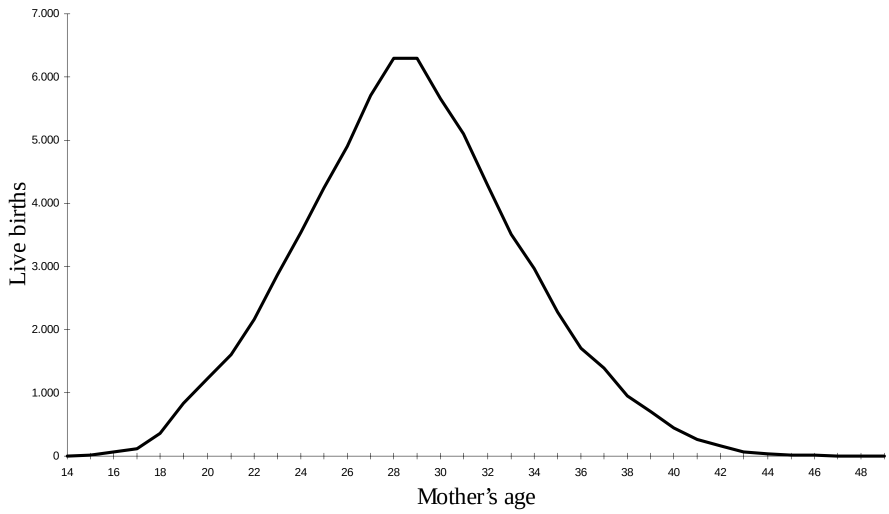

Figure 5 Distribution of mothers age for newborns in 1995

Note that figure 5 includes mothers younger than 18 when giving birth. In the model this possibility is ignored because of the assumption that all mothers are minimum 18 years old. This assumption is made for convenience and is unproblematic, since it is only the case for less than 0,3% of all births that the mother is below 18 years.

Thus figure 5 gives the first row of Ψ_{1995}. Assuming this maternal-distribution in 1995 to be time-invariant subsequently $\psi_{c,b,t} \equiv \psi_{c,b}$ for all c, b and all $t \geq 1995$[13]. Further assume that the mortality rate of children is independent of the age of their mother, then $\psi_{c,b} = \psi_{c+1,b+1}$ for all c, b. This means that all rows in $\Psi_t \equiv \Psi$ are identical, except for being displaced diagonally.

[13]This distribution, however, is not time invariant. This is implied by the fact that in 1982 the average-age of the first-time mother was 24,4 years, where it in 1992 was 26,4 years (Statistics Denmark; 1996, table 55).

The number of children that belong to each household (defined by maternal age), is determined by matrix multiplication, i.e. $\vec{N}_t^C \times \Psi = \vec{N}_t^{CF}$. Thus, the resulting vector $\vec{N}_t^{CF} \in R^{60}$ is the number of children defined by the age of their mother (or household), instead of their own, measured in adult-equivalents. An element in the $\vec{N}_t^{CF}$ vector, e.g. the element $N_{b,t}^{CF}$, which measures the amount of children belonging to households of age b may be written as

$$N_{b,t}^{CF} = \sum_{c=0}^{17} N_{c,t}^{C} \cdot \psi_{c,b} \tag{3.4}$$

where $N_{c,t}^C$ denotes the c'th element in the $\vec{N}_t^C$-vector (i.e. children c years old at time t).

The total number of adult-equivalents of the households can now be found by adding the 3 vectors measuring the number of women per generation, $\vec{N}_t^F$, the number of men assigned to women of each generation, $\vec{N}_t^{MF}$ and the number of children (measured in adult-equivalents) assigned to the generations of their mothers, $\vec{N}_t^{CF}$, so

$$\vec{N}_t^{EF} = \vec{N}_t^{F} + \vec{N}_t^{MF} + \vec{N}_t^{CF} = \vec{N}_t^{AF} + \vec{N}_t^{CF} \tag{3.5}$$

where $\vec{N}_t^{EF}$ is a vector, where each element, $N_{b,t}^{EF}$ measures the number of adult-equivalents of a specific age b of households.

3.2 Forecasting the number of adult-equivalents

For each vintage of women we define a representative household. The number of adult-equivalent members in the household at time t, where the woman is of age b, is given by the adult-equivalents, $N_{b,t}^{EF}$ constructed above. This household finds the optimal distribution of the consumption bundle per adult-equivalent over time, and determines the optimal composition of the consumption bundle in each period of time. Finally, the representative household decides upon the optimal labor supply of the adult members of the household in each period of time. Due to an additivity assumption in the utility function, the labor supply decision does not involve intertemporal substitution effects. The distribution of the consumption bundle over time, depends *inter alia* on the expected size of the household measured in adult-equivalents. This size depends on how the population is forecasted and how the households are constructed.

3.2.1 Population forecast

The population is forecasted using a couple of exogenous demographic assumptions[14]. In the present version it is assumed that:

1. The age and gender specific death rates for men, women, and children are kept constant at the 1995 level. This means that figure 6 illustrated later is assumed constant over time.

2. The net migration rate is 0 for both genders of each generation for all t. This means that only births give an inflow to the population.

3. The fertility is constant (!) and *in*dependent of the number of women in the fertile age or age-specific fertility rates. It is assumed that the number of newborns is constant through time and the distribution across the two genders is assumed to be identical to the distribution in 1995. The number of newborns is set such that the resulting stationary population is equal to the population in 1995 (measured in adult-equivalents).

It is possible to relax these assumptions and use other population forecasts in the model. The last assumption is made in order to get a stationary population fast, that is after everyone alive in 1995 have died; after 100 years. If fertility depended on the number of women in the fertile age and age-specific fertility rates, this would create echo effects in the population, that possibly would never die out - and thus the model would never arrive at its steady state. The assumption about migration is also made for practical purposes. In present Danish population forecasts the net migration is set to 13.500 persons annually (Statistics Denmark; 1996b) - but it is clearly unreasonable to assume that the migration continues at this rate for many centuries. Thus, a migration of zero is not chosen because it is believed to be realistic, but rather because it is an obvious focal point.

With the assumptions made above it is easy to forecast the population. The size of a given generation and gender at time $t+1$ is found from the size at time t adjusted for deaths using the age and gender specific survival rates. These survival rates are illustrated in figure 6. The figure shows the share of survivors there will be from a generation of newborns in 1995 at different levels of age.

[14] The way the population is forecasted is explained thoroughly in Appendix D.

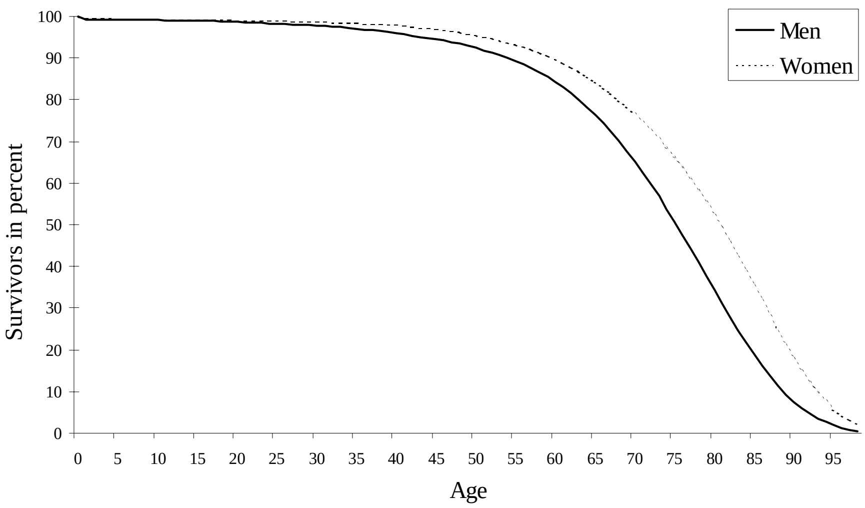

Figure 6 Survival rates for a generation born in 1995

Note that in the model it is not possible to be older than 99 years. This is equivalent to assuming that the survival rate at age 99 is 0.

3.2.2 Household forecast

In order to construct synthetic households from the population forecasted above, some assumptions are needed:

1. The normalized couple matrix $\bar{\Omega}_t$ is constant through time, i.e. $\bar{\Omega}_t = \bar{\Omega}$ for all t (as mentioned earlier).

2. The maternal matrix is constant through time, i.e. $\Psi_t = \Psi$ for all t (also assumed earlier). This means that figure 5 is unchanged over time.

Once again these assumptions can be replaced by others. It is probably not reasonable to assume that these matrices are constant over time - but coming up with alternative specifications of these matrices would be pure speculation.

It is not possible to forecast the household size at time $t+1$ only using the size at time t and taking deaths and births into account, since the distribution of men attached to the household differ as the age of the household differ, implying that men moves around between household of different ages. Thus forecasting the household size is a sum of the forecasts made for the members of the household.

Determining the adult members of the household ($\vec{N}^{AF}_{t+1}$) at time $t+1$ is done by distributing the amount of 17-year-old's, adult men and adult women surviving from time t according to the couple-matrix, $\bar{\Omega}$, to create households in the manner previously described. The forecasted number of children ($\vec{N}^{CF}_{t+1}$) is converted to adult-equivalents and these are assigned to their mother using the maternal matrix, Ψ. Adding these two gives the total number of adult-equivalents in each representative household; $\vec{N}^{EF}_{t+1} = \vec{N}^{AF}_{t+1} + \vec{N}^{CF}_{t+1}$.

3.3 Description of the constructed data-series

Figure 7 displays the average difference in age between the man and the woman in a couple, as a function of the age of the woman (as well as the standard deviation). One observes a relatively large variation in the difference in age for women at different ages. Whether these fluctuations are indeed age specific or generation specific, has not been investigated empirically. The assumption presented above implies that the entire fluctuation is assumed to be age specific.

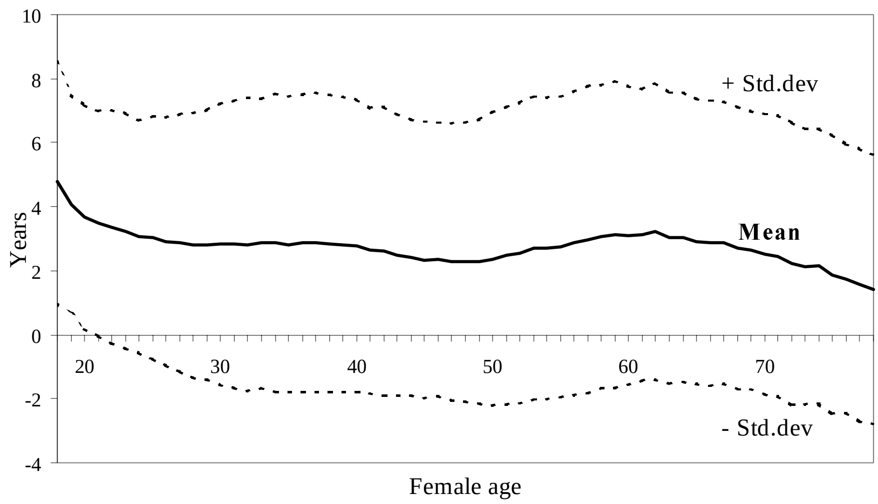

Figure 7 Differences in age for couples

Figure 8 shows the absolute age distribution of the two genders in 1995.The thin full line is age distribution of men measured according to their own age, the dotted line is the age distribution of women, and the thick full line is the age distribution of men measured according to the age of their partner. This curve is constructed using the normalized couple-matrix, $\bar{\Omega}$ as described in the previous subsection. Finally, the

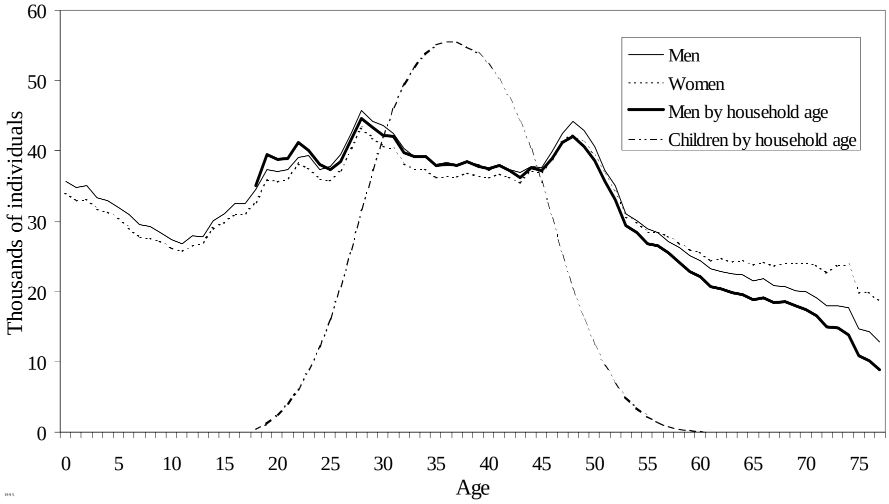

Figure 8 Age distribution of the population in 1995

hump-shaped curve is the age distribution of children according to the age of their mother.

The figures 9 and 10 display the predicted evolution in the population, given the assumptions listed in the previous subsection. Figure 9 shows the predicted distributions after 30 years.

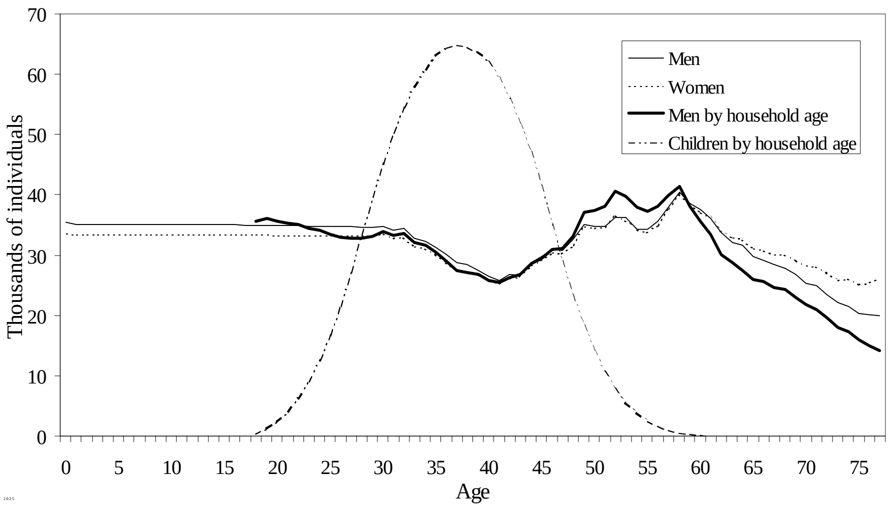

Figure 9 Age distribution of the population in 2025

The figure shows that the assumption of a constant number of newborns in the forecasting period, implies that the first part of the age distribution for both men and women becomes smooth, as the number of persons of both genders are affected only by the age and gender specific mortality rates (as shown in figure 6). Thus the assumptions imply, that all echo effects in the population are ignored. However, the mapping of the age distribution of men into the age distribution of their partners remains uneven due to fluctuations in the age difference of couple through the life cycle.

Figure 10 shows the stationary population. These age distributions will appear after 100 periods due to the assumption of constant fertility. Observe that even in the situation with a stationary age distribution of both genders, there will be an fluctuations in the number of men distributed according to the age of their partner.

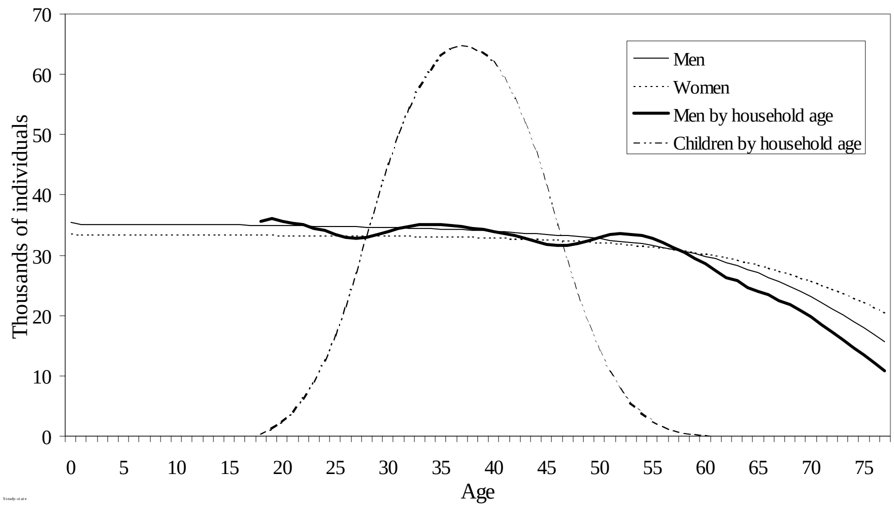

Figure 10 Age distribution of the population in the stationary state

As households are expected to have perfect foresight, the calculated changes in the size of the representative household are used for two purposes: 1) to find the optimal distribution of consumption per adult-equivalent in the solution to the maximization problem of the household, and 2) to generate the macro consumption from the consumption per adult-equivalent for each generation. The latter implies that a large generation has a large share in total consumption.

The notion of a representative household should be stressed here. The behavior of the households in the model implies that the household foresees the changes in its own age structure over time according to the normalized couple matrix. Literally, this means that the household foresees the divorces and re-marriages throughout the time horizon.

The household maximizes welfare over the deterministic finite time horizon. The interpretation of this is, that although individuals die and thus reduce the size of the household, the household as such does not cease to exist before the fixed time horizon. When the households reach the age, at which their time horizon expires (when the woman turns 78), they leave a foreseen bequest for younger generations.

3.4 Implications of incorporating the demographic structure

Before turning towards the optimal behavior of the households, it is relevant to consider how the incorporation of the demographic structure affects the expected size of households through the life-cycle. As a starting point, consider first the expected size of the household in the Auerbach and Kotlikoff (1987) model and the Blanchard (1985) model respectively. Auerbach and Kotlikoff assume "sudden death" and their decision units consist of individual agents. This implies that the expected size of the household is constant through the life-cycle; this is illustrated in the left part of figure 11. Contrary to this, in the Blanchard model the agent faces a constant probability of death, thus in that model the expected size of the household is exponentially decreasing through the life-cycle as shown on the right-hand of figure 11.

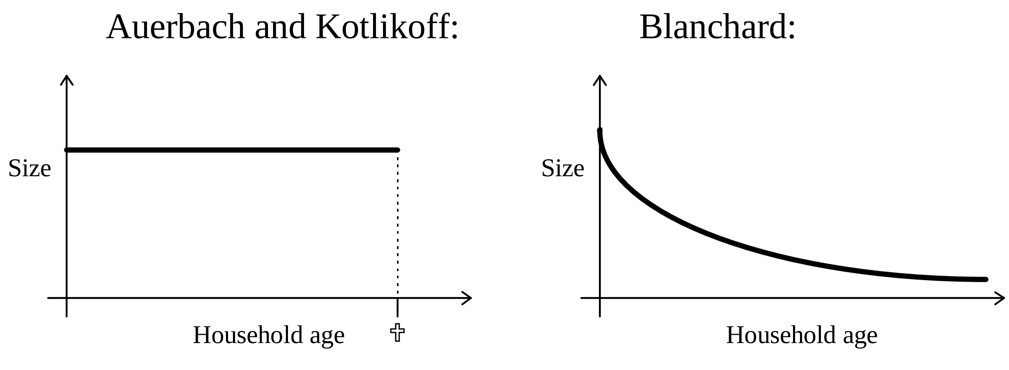

Figure 11 Households in the Auerbach & Kotlikoff and the Blanchard model

Comparison to figure 10, which displayed the distribution of the stationary population, reveals that the expected size of a given generation (if the age specific death rates are assumed constant through time) looks like a combination of the assumptions in the two models, such that for generations younger than 50 years the annual mortality rate is close to 0. After a relative short phase of life where the age specific mortality rate increases, the mortality rate for the remaining lifetime can be approximated by a constant as in the Blanchard model.

In conclusion, none of the two models alone give a realistic description of the expected size of a given generation of agents. This conclusion is strengthened, once one considers other demographic factors affecting the size of the household. First of all, the expected size of a representative household must be increasing in the period of life, where children are born into the household and similarly decrease in size in the period of life where the children leave home and form households of their own. Finally, the expected size of the household may be affected by the fact, that the expected difference in age between the persons in the couple is not constant throughout the life cycle (through divorces and re-marriages).

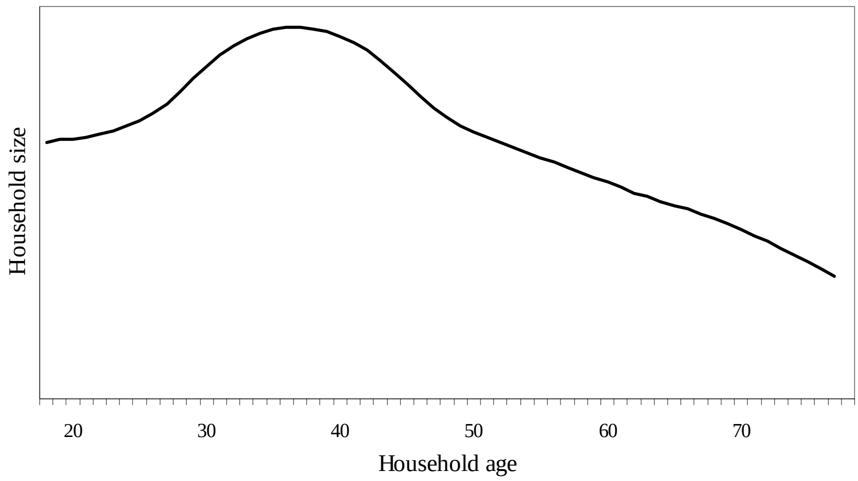

Figure 12 Household structure in the present model

Figure 12 gives the household structure in the present model and is comparable to the expected size of the household in the Auerbach and Kotlikoff model and the Blanchard model (figure 11). In general, of course, the savings decision of the household depends on the expected future size of the household. Thus the savings behavior of the households in Auerbach and Kotlikoff's and Blanchard's model differ from the savings behavior in the present model, as a result of the different assumptions of the expected size of the representative household.

Assuming that the utility function exhibits constant intertemporal elasticity of substitution (which is the most common assumption in CGE-models and also the one applied here), and that the intertemporal elasticity of substitution is less than 1, implies that agents in the Blanchard-model *ceteris paribus* will save less than agents in the Auerbach-Kotlikoff model. The opposite is true if the elasticity of substitution is larger than 1. In the present model savings are affected both by the presence of children and by the fact that the mortality rate is positive. Except for the very young households, we would *ceteris paribus* expect that savings are lower in households with this age structure than in both the Blanchard model and the Auerbach and Kotlikoff model.

The major difference in the macroeconomic results between, the three different ways of modelling the expected size of the household appears if the age structure of the population is not stationary, i.e. if one analyzes e.g. the consequences of the aging problem. To illustrate the difference, consider a decrease in the birth rate. In the present model, this implies that less children are expected to be born by the representative household. Therefore the expected size of the household becomes less "peaked". Thus at the beginning of their life, the households' savings will tend to be lower, whereas later on savings will tend to be larger.

In this way a reduction in the birth rate, implies both that a larger part of the households become older (in the phase until the population is adjusted to the lower stationary level), and that the behavior of the young households are different from the behavior that the old households had when they were young.

Analyses of this phenomenon using the Auerbach and Kotlikoff model would imply that only the first of the effects mentioned above would be present.[15] Thus the explicit modelling of the demographic structure adds a qualitatively different effect into the analysis of changes in the composition of the population.

To sum up, it is obvious that the household structure presented here has a distinct advantage over the traditional methods. The two traditional methods may be convenient, but for most purposes they can hardly be considered realistic. It is tempting to label the method presented here "Full Information" since it uses all available data, and makes very few assumptions for convenience.

[15] The Blanchard model is not well suited for this experiment, as the model presupposes that the population is constant, through the assumption that the birth rate is identical to the death rate.

Chapter 4
HOUSEHOLDS

The previous chapter described how households were constructed - the purpose of this chapter is to describe their economic behavior and decision making. The problem faced by the consumers is indeed a classical one; maximizing lifetime utility subject to a budget constraint. The intertemporal utility function is the sum of discounted instantaneous (one-period) utilities for each period in the lifecycle plus utility from leaving a bequest.

To solve the household's problem we need first to determine the budget constraint, i.e. specify the stream of income for an adult belonging to a household of a specific age at a given time; this is done in the first section. The next section outlines this utility function that the household seeks to maximize. The third section describes the intertemporal optimization problem, and it is shown that this problem can be decomposed into an optimal labor supply decision and an optimal consumption decision. Section four discusses the bequest decision, i.e. how the household determines the size of the bequest to leave behind, and how it is transferred in to inheritance for the recipients. Finally section five discusses the consumer's one-period decision problem, i.e. how to compose total consumption in a given period (the solution to the intratemporal problem) of the different goods in the economy.

4.1 Non-interest income

The income of an individual in a given period of time depends on whether he or she is working or has retired from the labor market and has become a pensioner. To simplify, the retiring decision is assumed to be exogenous and the retirement age is set to 61.

Before proceeding it is worth noting that **all agents of a specific gender and age are assumed to be identical**. This means, that the recorded level of gender specific unemployment is assumed to be shared between all persons of a given gender in the generation[1]. Therefore underemployment might be a more appropriate term. A similar assumption applies to the number of working hours, inheritance and public transfers.

Each individual receives age and gender specific transfers from the public sector.[2] There are two kinds of age specific transfers: The tax free transfers, $TR^J_{i,t}$, $J = F,\ M$, and the taxable transfers $TRT^J_{i,t}$, $J = F,\ M$ of which the tax $T^T_{i,t}(TRT^J_{i,t})$ is paid. Also each individual gets lump sum transfers, τ^W_t and τ_t, from abroad and the domestic government respectively. The general lump sum transfer, τ_t, from the government is used to balance the budget of the public sector.

For a working individual the non-interest income is in addition composed of a salary net of taxes, an unemployment benefit net of taxes, and inheritance. This amounts to

$$[W_t - T^w_t(W_t)]\,\ell^J_{i,t} + \left[b_t - T^b_t(b_t)\right]\left(\bar{\ell} - \ell^J_{i,t}\right) + B^{inJ}_{i,t} \quad J = F,\ M \tag{4.1}$$

where W_t is the wage rate at time t,[3] b_t is the level of unemployment benefits, $T^w_t(W_t)$ and $T^b_t(b_t)$ are the taxes paid of income from employment and unemployment benefits respectively. The specification of the tax function allows for progressive income taxes. $\bar{\ell}$ is an institutionally fixed maximum working time, such that e.g. $\left(\bar{\ell} - \ell^F_{b,t}\right)$ is the amount of hours that a woman of age b is unemployed (or more correctly underemployed) during period t. Finally, $B^{inJ}_{i,t}$, is the inheritance at time t received by gender J aged i. Note, that the specification of the distribution of births and the assumption that households survives until the woman reaches the age of 78 years, imply that heirs will have at maximum age of $(78 - 18 =)$ 60 years, because the youngest mothers are 18 years old. Similarly, the minimum age of a heir is $(78 - 49 =)$ 29 years, as the maximum age of mothers giving birth is 49 years. This implies that $B^{inJ}_{i,t}$ is positive only for $i \in [29, 60]$.

For a person who is retired from the labor market the only non-interest income source

[1] Note however that some men, who are less than 60 years old may be married to women, who are older than 77 years and therefore these men are defined as economically inactive.

[2] Presently, the levels of age specific public transfers are indexed to the wage net of labor market contributions. Later this feature and the assumption of exogenous retirement will be treated more elaborately. Age specific transfers are calculated from the project of "generational accounting" undertaken jointly by the Ministry of Finance and EPRU. These transfers are part of the input data for the model.

[3] Observe that the wage is assumed to be identical across generations and gender, implying that labor productivity is the same for all generations and both genders.

(aside from eventual age specific transfers) is the public pension, f_t^P, which is being taxed according to the tax-function $T_t^P\left(f_t^P\right)$. We define the social security pension as age independent. This implies that we ignore differences between the standard social security pension (applying to all persons of 67 or older) and the labor market related pension scheme ("efterlønsordningen") which in principle is not a general pension scheme.

Further let $N_{b,t}^{FW}$ denote the number of female workers aged b at time t

$$N_{b,t}^{FW} \equiv \begin{cases} N_{b,t}^{F} & for\ 18 \leq b < 61 \\ 0 & for\ b \geq 61 \end{cases} \tag{4.2}$$

We can now write the total non-interest income per adult of a household aged b at time t, $y_{b,t}$, as below

$$\begin{aligned} y_{b,t} \ = \ & \frac{1}{N_{b,t}^{AF}} \Big\{ N_{b,t}^{FW}\left(\left[W_t - T_t^w\left(W_t\right)\right]\ell_{b,t}^F + \left[b_t - T_t^b\left(b_t\right)\right]\left(\bar{\ell} - \ell_{b,t}^F\right) + B_{b,t}^{inF}\right) \\ & + \left(N_{b,t}^F - N_{b,t}^{FW}\right)\left[f_t^P - T_t^P\left(f_t^P\right)\right] \\ & + N_{b,t}^F \cdot \left(TR_{b,t}^F + \left[TRT_{b,t}^F - T_{b,t}^T\left(TRT_{b,t}^F\right)\right]\right) \\ & + \left[W_t - T_t^w\left(W_t\right)\right]\sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \cdot \ell_{a,b,t}^M \\ & + \left[b_t - T_t^b\left(b_t\right)\right]\sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \cdot \left(\bar{\ell} - \ell_{a,b,t}^M\right) \\ & + \sum_{a=18}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} \cdot \left(TR_{a,t}^M + \left[TRT_{a,t}^M - T_{a,t}^T\left(TRT_{a,t}^M\right)\right]\right) \\ & + \sum_{a=29}^{60} N_{a,t}^M \cdot \omega_{a,b} \cdot B_{a,t}^{inM} + \left[f_t^P - T_t^P\left(f_t^P\right)\right]\sum_{a=61}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} \Big\} + \tau_t^W + \tau_t \end{aligned} \tag{4.3}$$

To get the total household income provided by women, the sum of income from employment and unemployment benefits, both net of taxes, and the inheritance (i.e. the 3 terms in the brackets in the first line) is multiplied by the number of female workers in the representative household, $N_{b,t}^{FW}$. The pension after tax is multiplied by the number of female pensioners. The taxfree and taxable age specific public transfer are both simply weighted by the (total) number of women in the household.

For households where the women are pensioners ($b \geq 61$), the first line of expression (4.3) cancels out because the number of female workers equals zero. Correspondingly the term $N_{b,t}^F - N_{b,t}^{FW}$ in the second line reduces to $N_{b,t}^F$. On the other hand in the case of $18 \leq b < 61$, this last term cancels out (i.e. no pension) while the first line remains.

The first two lines (inside the braces) represents the total non-interest income after tax of women in the household whereas the rest of the expression inside the braces relates to the income of men in the household. This part of the expression becomes more complicated, as the men in the representative household do not share the same age. Some men are retired, whereas others are not.

Therefore the men in the household - regardless of the age of the woman - contribute both with the income terms related to workers and to pensioners. The total income arising from men in the household is found by multiplying the number of men in each age group with their respective income terms and then summarize these expressions over the relevant interval of ages. Income from employment is e.g. given by wages per hour after tax, $[W_t - T_t^w(W_t)]$ (which is the same for all age groups), times the male labor supply of the household. This is given by $\sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \cdot \ell_{a,b,t}^M$ (for the household of age b). Note that there is added a b in the subscript for the male labor supply. This is done to distinguish between the supply of labor for men at a given age married to women of different ages.

To measure per adult we divide the sum of income from men and women in the household, by the number of adults, $N_{b,t}^{AF}$. To this we add the two lump sum transfers, τ_t^W and τ_t, which are defined per adult, and we finally obtain the stream of non-interest income per adult belonging to a household of a specific age at a given moment in time.

To forecast the social security pensions, the unemployment benefits and the age specific transfers, we endogenize these transfers by relating them to the level of wages. The pre-tax social security pension is indexed to the wage rate net of labor market contributions such that

$$f_t^P = \varphi^P W_t \left(1 - t_t^\ell\right) \tag{4.4}$$

where t_t^ℓ is the labor market contribution tax rate. Nominal unemployment benefits in Denmark are indexed to the overall wage index, W_t with a lag of approximately two years. In the model this lag is ignored and the benefits are regulated instantly according to

$$b_t = \varphi^b W_t \left(1 - t_t^\ell\right) \tag{4.5}$$

where $0 < \varphi^b < 1$. The tax free age specific transfers are indexed as

$$TR_t^J = \varphi^J W_t \left(1 - t_t^\ell\right) \quad J = F,\, M \tag{4.6}$$

and the taxable age specific transfers are indexed analogously

$$TRT_t^J = \varphi^{TJ} W_t \left(1 - t_t^\ell\right) \quad J = F,\, M \tag{4.7}$$

Observe from (4.3) that the taxation of wages is assumed to be imposed directly on wages per hour. This (unrealistic) feature is introduced to compensate for unintended effects of the assumption of work-sharing. The latter implies, that the amount of unemployment is artificially converted into a reduced number of working hours per employed and therefore higher unemployment implies lower annual pre-tax wages, which would have reduced the marginal tax rate, had a conventional progressive tax system been employed. This artificially lower marginal tax rate for employed workers affects individual behavior, which in turn would alter the equilibrium. To avoid this, we have chosen the specification above, to get a tax system which has an effect on individual behavior, that can resemble the effects of a standard progressive tax system in an economy, where unemployment is unequally distributed between persons.

We now turn to specify the preferences of the household.

4.2 The household's utility function

Consider a specific generation of the representative households aged b at a given point in time t. This means, that the household is born in the year $t - b$. At the beginning of year t,[4] where the women in the household turn b years, the rest-of-life discounted utility for the representative household in question, $U_{b-1,t-1}$, is given by a CES-function

$$U_{b-1,t-1} \equiv \left[\sum_{i=b}^{78} \left(Q_{i,t-b+i}\right)^{\frac{S-1}{S}} v_{b-1,i} \cdot N_{i,t-b+i}^{EF}\right]^{\frac{S}{S-1}} \tag{4.8}$$

$$v_{b-1,i} \equiv \xi_i \left(\frac{1}{1+\theta}\right)^{i-b+1} \text{ where } \xi_i = 1 \text{ for } i = b, ..., 77, \text{ and } \xi_{78} = \xi \tag{4.9}$$

The utility, (4.8), is defined as the sum of annual total household utilities (equal to instantaneous utilities per adult-equivalent, $(Q_{b,t})^{\frac{S-1}{S}}$, times the number of adult-equivalents in the household, $N_{b,t}^{EF}$) in the remaining lifetime (from age b to age 77),

[4] Observe from the dating conventions of the model (see chapter 1), that households enter the economy in the begining of the period. Therefore the utility is discounted back to this point in time. In the following the beginning of period t is approxated by the end of period $t - 1$.

discounted to the beginning of year t. In addition to this sum of annual utilities, the household obtains positive utility from leaving a bequest to the children. This additional utility is measured as an extra period of (pure) consumption (period 78) differentiated only from normal consumption by a preference parameter ξ.

$Q_{78,t-b+78}$ is the real value (as measured by the price index of consumption) of the bequest per adult(-equivalent) that the generation leaves to future generations at the time when it reaches 78 years of age and the household ceases to exist, ξ is the weight that generations associates with the bequest.

Note that the criterion applied in the objective function is the Benthamite (after Jeremy Bentham, 1748-1832) as opposed to the Millian criterion (after John Stuart Mill, 1806-1873) used in previous versions of this paper. Thereby the objective is shifted from maximizing average utility to maximizing total utility - known as classical utilitarianism. The multiplication by $N_{b,t}^{EF}$ in the utility function (which is the new element) implies that households in the planning of the optimal consumption over time will take into account the fluctuations in household size, not only in the budget constraint, but also in their preferences by the weights put on instantaneous utilities at different points in time. In this way utility stemming from consumption at the end of the planning period will not be weighted very highly, since only a small part of the household members will live to enjoy it. On the other hand much weight will be put on utility per head in the mid-thirties where the household reach its maximum size[5].

4.2.1 How to model utility from leaving a bequest?

The modelling of the bequest decision may in general take two very different routes. First, bequest may be **altruistic** as introduced by the seminal article by Barro (1974). This implies that the present household takes the utility of future generations directly into account by including the utility functions of these generations in the object function of the household's maximization problem. This leads to a model of "dynasties" where each dynasty maximizes utility over an infinite horizon. Assuming that no new dynasties enter the economy, this implies that the overlapping generation structure collapses and the model converts into a Ramsey type model. In this type of model the gradual dynamic transition of the economy, which is due to the entry of new house-

[5] For a more thorough presentation of the two criteria and an analysis of the implications of the choice between them, see Knudsen and Pedersen (1998).

holds in the overlapping generations model, is replaced by intergenerational transfers within each dynasty.

Although intergenerational transfers do play a role in most families, it appears that such transfers are less perfect than is the case given by the altruistic assumption. Therefore we have in this model chosen to model a "**joy of giving**" motive for bequest. In this case the parent household obtains utility simply by leaving a gift to its heirs. The effect on the utility of the heirs does not directly enter into bequest decision of the parent household. Therefore e.g. the number of heirs is seen not to affect the bequest decision in the formulation given in equation (4.8). This implies that the overlapping generation structure of the model is maintained although some imperfect intergenerational transfers takes place.

Observe, that the "joy-of giving" motive for bequest implies that bequests has a "double" effect on the utility of the agents as both the donor and the receiver obtains utility from the same transfer. Bequest is the only kind of transfer in the economy which exhibits the "double utility" feature, therefore the utility effects of policy experiments, which increases this transfer on the expense of other transfers may be overstated.

4.2.2 The instantaneous utility function

The annual utility function exhibits constant intertemporal elasticity of substitution. This elasticity is denoted by $S > 0$. θ is the rate of pure time preference and $v_{b-1,i}$ is the discount factor from period i to period $b-1$. $Q_{b,t}$ is the instantaneous utility of the household.

The instantaneous utility function, (4.10), is defined as the real value of the consumption bundle per adult-equivalent, $C_{b,t}$ minus the disutility of work for the females and the males measured per adult-equivalent, $Z_{b,t}$ in the representative household respectively

$$Q_{b,t} = C_{b,t} - Z_{b,t} \tag{4.10}$$

The consumption index $C_{b,t}$ is a CES index defined over domestic and imported consumption goods and the disutility of work $Z_{b,t}$ is given by

$$Z_{b,t} \equiv \begin{cases} \frac{N_{b,t}^{FW}}{N_{b,t}^{EF}} \cdot f\left(\ell_{b,t}^{F}\right) + \frac{1}{N_{b,t}^{EF}} \sum\limits_{a=18}^{60} N_{a,t}^{M} \cdot \omega_{a,b} \cdot f\left(\ell_{a,b,t}^{M}\right) & \text{for} \quad 18 \leq b < 78 \\ 0 & \text{for} \quad b = 78 \end{cases} \tag{4.11}$$

where $\ell_{b,t}^{F}$ is the number of working hours per woman at age b, and $\ell_{a,b,t}^{M}$ is the number of working hours per man aged a in the household aged b. $f(\cdot)$ is the disutility of work. Thus, $N_{b,t}^{FW} \cdot f\left(\ell_{b,t}^{F}\right)$ is the sum of the disutility of work for women in the representative household at age b. Similarly, $\sum_{a=18}^{60} N_{a,t}^{M} \cdot \omega_{a,b} \cdot f\left(\ell_{a,b,t}^{M}\right)$ is the sum of the disutility of work for the men in the b-years old representative household. Remember that only the part of the adults who are younger than 61 have a labor supply. Dividing both expressions by the number of adult-equivalents in the household at the time, $N_{b,t}^{EF}$ implies that total disutility of work is measured per adult-equivalent.

4.3 Intertemporal optimization

The maximization problem of the household may in principle be solved using the same approach as the decision problem of the firms (see chapter 2). In this case, one has to set up the Hamiltonian using the stock of wealth per adult equivalent as the state variable, and maximize this function with respect to the control variables, which are the household's demand for each of the two consumption goods and the household's supply of labor for both genders and all possible ages of the distribution of men in the household.

The specification of the utility function (4.8) as a CES function over instantaneous utility index, $Q_{b,t}$, and the definition of the instantaneous utility (4.10) as a CES function over two additive terms (i.e. consumption goods and disutility from labor), implies that the solution may (easier) be found using the general solution to a CES function, which in a general form is presented in Appendix B.

The solution of the household's maximization problem follows a two step procedure. In the first step, the household decides on the optimal evolution of the level of the consumption index, $C_{b,t}$, and the labor supply through time, i.e. the household determines the intertemporal evolution of the instantaneous utility index, $Q_{b,t}$. In the second step, the value of the consumption index at each moment of time is split into its components.

The intertemporal maximization problem for the representative household of age b at time t , is to maximize rest-of-life discounted utility given by (4.8), using the definitions of $Q_{b,t}$ and $Z_{b,t}$ ((4.10) and (4.11)), and subject to the conditions

$$a_{b,t} = (1 + r_t (1 - t_t^r)) a_{b-1,t-1} \frac{N_{b-1,t-1}^{EF}}{N_{b,t}^{EF}} + y_{b,t} \frac{N_{b,t}^{AF}}{N_{b,t}^{EF}} - P_t C_{b,t} \tag{4.12}$$

$$a_{78,t-b+78} = 0 \tag{4.13}$$

$$a_{b-1,t-1} \text{ is given (if } b \leq 18 \text{ then } a_{b-1,t-1} = 0) \tag{4.14}$$

where $a_{b,t}$ is the stock of wealth per adult-equivalent of household b at time t. Recall that financial assets consist of both shares and bonds. Arbitrage implies that the tax adjusted yields from the two types of assets are identical in every period, and therefore we need only to consider the evolution of the total stock of wealth and not the distributions across assets. Condition (4.12) is the savings identity measured per adult-equivalent in the household. The first term on the right hand side, is last period's stock of wealth, plus interest income per adult-equivalent in the present period. The second term is the non-interest income, $y_{b,t}$, given in (4.3). Recall that $y_{b,t}$ is measured per adult. To find the income per adult-equivalent we have to multiply by the fraction $N_{b,t}^{AF}/N_{b,t}^{EF}$. The last term, is the cost of the consumption bundle per adult equivalent, where P_t is the price index associated with the consumption index, $C_{b,t}$.

Condition (4.13) is the terminal condition, that the stock of wealth which the household holds at the end of its life time (age 77), is equal to the bequest which it leaves to its successors. This implies that the stock of wealth is zero at "age" 77+1. Condition (4.14) states that the initial stock of wealth is given by the past, and that it is zero if the household is of age 18.

To solve the optimization problem, first rewrite (4.12)

$$a_{b,t} = (1 + \tilde{r}_{b,t}) a_{b-1,t-1} + \tilde{y}_{b,t} - P_t Q_{b,t} \tag{4.15}$$

where

$$\tilde{r}_{b,t} \equiv (1 + r_t (1 - t_t^r)) \frac{N_{b-1,t-1}^{EF}}{N_{b,t}^{EF}} - 1 , \tag{4.16}$$

$$\tilde{y}_{b,t} \equiv y_{b,t} \frac{N_{b,t}^{AF}}{N_{b,t}^{EF}} - P_t Z_{b,t} \tag{4.17}$$

Thus $\tilde{r}_{b,t}$ is the after tax interest rate adjusted for the growth in the number of persons in the household (measured in adult-equivalents) and $\tilde{y}_{b,t}$ is the non-interest income per adult-equivalent corrected for the disutility value of work. Thus given the tax system and the specified function for the disutility of work, $\tilde{y}_{b,t}$, measures the net gain from working (measured in units of the foreign good) per adult in the household.

4.3.1 Optimal labor supply

Not surprisingly, $\ell^F_{b,t}$ and $\ell^M_{a,b,t}$ should be chosen such that $\tilde{y}_{b,t}$ is maximized at all times. Intuitively, this maximizes the net gain from work in the household. Technically it follows from (4.15) and (4.8) that $\tilde{y}_{b,t}$ should be maximized at all times: Since a higher value of $\tilde{y}_{b,t}$ expands the feasible value of the instantaneous utility index, $Q_{b,t}$, it must be utility-maximizing to optimize $\tilde{y}_{b,t}$ with respect to the labor supplies.

From (4.17) and also using (4.11), the first order conditions are

$$\frac{\partial \tilde{y}_{b,t}}{\partial \ell^F_{b,t}} = 0 \quad \Leftrightarrow \quad N^{FW}_{b,t} f'\left(\ell^F_{b,t}\right) = \frac{1}{P_t} N^{AF}_{b,t} \frac{\partial y_{b,t}}{\partial \ell^F_{b,t}} \tag{4.18}$$

$$\begin{aligned} \frac{\partial \tilde{y}_{b,t}}{\partial \ell^M_{a,b,t}} &= 0\,,\ \forall\ a \in [18;60]\ \Leftrightarrow \\ N^M_{a,t} \cdot \omega_{a,b} \cdot f'\left(\ell^M_{a,b,t}\right) &= \frac{1}{P_t} N^{AF}_{b,t} \frac{\partial y_{b,t}}{\partial \ell^M_{a,b,t}}\,,\ \forall\ a \in [18;60] \end{aligned} \tag{4.19}$$

The interpretation of the optimality condition (4.18) is simply that the women of the household shall work up to the point where the disutility of an extra unit of work times the number of female workers is exactly offset by an equal rise in the real non-interest income multiplied by the number of adults in the household. The condition (4.19) can be interpreted in a similar way. For each generation of men the increase in the disutility of the household from an extra unit of work must equal the gain in real income for the household. Since the condition has to be satisfied for all ages of men between 18 and 60, the expressions (4.18) and (4.19) together represent 44 first order conditions.

From the definition of non-interest income, $y_{b,t}$, given in expression (4.3) the partial derivatives of this can be deduced

$$\frac{\partial y_{b,t}}{\partial \ell^F_{b,t}} = \frac{N^{FW}_{b,t}}{N^{AF}_{b,t}} \left(W_t - T^w_t\left(W_t\right) - \left[b_t - T^b_t\left(b_t\right)\right]\right) \tag{4.20}$$

$$\frac{\partial y_{b,t}}{\partial \ell^M_{a,b,t}} = \left(W_t - T^w_t\left(W_t\right) - \left[b_t - T^b_t\left(b_t\right)\right]\right) \frac{N^M_{a,t} \cdot \omega_{a,b}}{N^{AF}_{b,t}} \tag{4.21}$$

Subsequently define the function for disutility of work as

$$f\left(\ell^J_{i,t}\right) = \left(\frac{\gamma}{\gamma+1}\right) \gamma_1 \left(\ell^J_{i,t}\right)^{\frac{\gamma+1}{\gamma}}\,,\quad J = F,\, M \tag{4.22}$$

Note that the disutility function is the same for men and women and the specification implies that the marginal disutility is increasing in the amount of work. Then by using this function and inserting the partial income derivatives, the first order conditions can be rewritten as

$$N_{b,t}^{FW}\gamma_1 \left(\ell_{b,t}^{F}\right)^{\frac{1}{\gamma}} = \frac{N_{b,t}^{FW}}{P_t}\left(W_t - T_t^w\left(W_t\right) - \left[b_t - T_t^b\left(b_t\right)\right]\right) \tag{4.23}$$

$$N_{a,t}^{M}\cdot\omega_{a,b}\cdot\gamma_1 \left(\ell_{a,b,t}^{M}\right)^{\frac{1}{\gamma}} = \frac{N_{a,t}^{M}\cdot\omega_{a,b}}{P_t}\left(W_t - T_t^w\left(W_t\right) - \left[b_t - T_t^b\left(b_t\right)\right]\right) \tag{4.24}$$

Finally solving for the labor supply of women and men respectively yields

$$\begin{aligned} \ell_{b,t}^{F} &= \begin{cases} \ell_t & \text{for} \quad 18 \leq b < 61 \\ 0 & \text{for} \quad b < 18 \vee b \geq 61 \end{cases} \\ &\text{and} \\ \ell_{a,b,t}^{M} &= \begin{cases} \ell_t & \text{for} \quad 18 \leq a < 61 \\ 0 & \text{for} \quad a < 18 \vee a \geq 61 \end{cases} \end{aligned} \tag{4.25}$$

where

$$\ell_t \equiv \left(\frac{\left[W_t - T_t^w\left(W_t\right)\right] - \left[b_t - T_t^b\left(b_t\right)\right]}{\gamma_1 P_t}\right)^{\gamma} \tag{4.26}$$

Observe first, that no intertemporal speculation in the labor supply occurs. This atemporal structure is due to the additivity assumption in the utility function, which excludes income and wealth effects from the labor supply decision. Thus labor supply in period t is a simple increasing function of the mark up of the wage after tax over the net of tax unemployment benefits in period t.

Secondly, both the female and the male labor supply is unaffected by the age of the person, as well as all other characteristics of the household (e.g. the number of children in the household). This lack of effects is due to the non-age specific formulation of the disutility of labor function, as well as the choice of normalizations of the disutility of labor in the utility function.

Finally, observe that the assumption that both the functional form and the parameters of the disutility function are identical across genders, imply that the labor supply of men and women is identical.

As mentioned the specification of the utility function will be subject to empirical investigations, which may significantly alter the simple an symmetric result presented here. The present formulation is chosen, since it represents the simplest possible benchmark.

4.3.2 Optimal consumption

Given the functions for the supply of labor (4.26), which is a function of prices and parameters all of which are exogenous to the household, we can now consider $\tilde{y}_{b,t}$ a known variable at all times. To solve the dynamic problem, means solving the differential equation (4.15). First define $\tilde{R}_{b,i}$ as the discount factor, discounting from period i to period b

$$\tilde{R}_{b,i} \equiv \prod_{j=b+1}^{i} \frac{1}{1+\tilde{r}_{j,t-b+j-1}} \ , \ for\ i > b \ \ and \quad \tilde{R}_{b,b} \equiv 1 \tag{4.27}$$

Using this discount factor we derive the consolidated budget constraint of the household using the savings identity (4.12) and the terminal condition for the stock of wealth (4.13). Observe that

$$a_{i,t-b+i}\tilde{R}_{b-1,i} - a_{i-1,t-b+i-1}\tilde{R}_{b-1,i-1} = (a_{i,t-b+i} - (1+\tilde{r}_{i,t-b+i})a_{i-1,t-b+i-1})\tilde{R}_{b-1,i}$$

From (4.15) this implies that

$$a_{i,t-b+i}\tilde{R}_{b-1,i} = a_{i-1,t-b+i-1}\tilde{R}_{b-1,i-1} + (\tilde{y}_{i,t-b+i} - P_{t-b+i}Q_{i,t-b+i})\tilde{R}_{b-1,i}$$

Therefore

$$\begin{aligned} a_{i,t-b+i}\tilde{R}_{b-1,i} &= a_{b-1,t-b+b-1}\tilde{R}_{b-1,b-1} + \sum_{j=b}^{i} \left(\tilde{y}_{j,t-b+j} - P_{t-b+j}Q_{j,t-b+j}\right)\tilde{R}_{b-1,j} \\ &= a_{b-1,t-1} + \sum_{j=b}^{i} \tilde{y}_{j,t-b+j}\tilde{R}_{b-1,j} - \sum_{j=b}^{i} P_{t-b+j}Q_{j,t-b+j}\tilde{R}_{b-1,j} \end{aligned}$$

This implies that

$$a_{78,t-b+78}\tilde{R}_{b-1,78} = a_{b-1,t-1} + \sum_{i=b}^{78} \tilde{y}_{i,t-b+i}\tilde{R}_{b-1,i} - \sum_{i=b}^{78} P_{t-b+i}Q_{i,t-b+i}\tilde{R}_{b-1,i} = 0$$

as $a_{78,t-b+78} = 0$, such that we derive at the consolidated budget constraint which may be written as

$$\sum_{i=b}^{78} \tilde{R}_{b-1,i}P_{t-b+i}Q_{i,t-b+i} = a_{b-1,t-1} + H_{b-1,t-1} \tag{4.28}$$

where

$$H_{b-1,t-1} \equiv \sum_{i=b}^{78} \tilde{y}_{i,t-b+i}\tilde{R}_{b-1,i} = \sum_{i=b}^{77} \tilde{y}_{i,t-b+i}\tilde{R}_{b-1,i} \tag{4.29}$$

because $\tilde{y}_{78,t-b+78} = 0$. $H_{b,t}$ denotes the human capital per adult-equivalent in the household of b years of age at time t.

The problem for the household is now to maximize the objective function given in (4.8) with respect to $Q_{i,t-b+i}$, $i = \{b; 78\}$, subject to the consolidated budget constraint (4.28).This is a standard CES-maximization problem. According to Appendix B, the solution to this problem is given by

$$Q_{i,t-b+i} = \left(v_{b-1,i}N^{EF}_{i,t-b+i}\right)^S \left(\frac{\tilde{R}_{b-1,i}P_{t-b+i}}{\eta_{b-1,t-1}}\right)^{-S} \frac{a_{b-1,t-1}+H_{b-1,t-1}}{\eta_{b-1,t-1}}, \; i \geq b \tag{4.30}$$

where

$$\eta_{b-1,t-1} \equiv \left[\sum_{i=b}^{78} \left(v_{b-1,i} \cdot N^{EF}_{i,t-b+i}\right)^S \left(P_{t-b+i}\tilde{R}_{b-1,i}\right)^{1-S}\right]^{\frac{1}{1-S}} \tag{4.31}$$

$\eta_{b,t}$ is a household specific CES price index of all future prices for the rest of the household's time horizon. Observe from (4.30) and the definitions of $v_{b-1,i}$, $\tilde{r}_{b,t}$ and $\tilde{R}_{b,i}$ - given by (4.9), (4.16) and (4.27) respectively - that

$$\begin{aligned}
Q_{b,t} &= \left(v_{b-1,b}N^{EF}_{b,t}\right)^S \left(\frac{\tilde{R}_{b-1,b}P_t}{\eta_{b-1,t-1}}\right)^{-S} \frac{a_{b-1,t-1}+H_{b-1,t-1}}{\eta_{b-1,t-1}} && (4.32)\\
&= \left(\frac{\xi_b}{1+\theta}N^{EF}_{b,t}\right)^S \left(\frac{\frac{1}{1+\tilde{r}_{b,t}}P_t}{\eta_{b-1,t-1}}\right)^{-S} \frac{a_{b-1,t-1}+H_{b-1,t-1}}{\eta_{b-1,t-1}} && (4.33)\\
&= \xi_b^S \left(\frac{1+\tilde{r}_{b,t}}{1+\theta}\frac{\eta_{b-1,t-1}}{P_t}N^{EF}_{b,t}\right)^S \frac{a_{b-1,t-1}+H_{b-1,t-1}}{\eta_{b-1,t-1}} && (4.34)\\
Q_{b,t} &= \xi_b^S \left(\frac{1+r_t\left(1-t^r_t\right)}{1+\theta}\frac{\eta_{b-1,t-1}}{P_t}N^{EF}_{b-1,t-1}\right)^S \frac{a_{b-1,t-1}+H_{b-1,t-1}}{\eta_{b-1,t-1}}, \quad 18 \leq b \leq 78 && (4.35)
\end{aligned}$$

This is the solution for the individual consumption net of disutility from work. Individual consumption depends on total individual wealth (the sum of financial wealth and human capital) divided by the CES price index. The effect on consumption of high future prices can be seen through a high level of this index. This reduces the real value of total wealth (for a given nominal value of the stock of human wealth, $H_{b-1,t-1}$) and thereby consumption. On the other hand an increase in future prices makes consumption today relatively cheaper and hereby increases the propensity to consume today. If the intertemporal elasticity of substitution, S, is greater than 1 the latter effect will dominate, whereas a low value of S (< 1) will mean that an increase in future prices will be associated with a fall in today's consumption - the income effect is then dominating[6].

The fact that current consumption depends on the sum of financial and human wealth, where the latter represents the discounted value of the future stream of non-interest income, is due to the joint assumption of perfect foresight and perfect capital markets, which enables the consumer to transform any stream of income into another stream of income with the same discounted value. This means that the consumer does not face constraints in the capital market, as long as he does not violate the consolidated budget constraint. Therefore future income may readily be transformed into current consumption.

From (4.30) we have

$$Q_{b+1,t+1} = \xi_{b+1}^{S} \left(\frac{1+r_t(1-t_t^r)}{1+\theta} \frac{1+r_{t+1}\left(1-t_{t+1}^r\right)}{1+\theta} \frac{\eta_{b-1,t-1}}{P_{t+1}} N_{b-1,t-1}^{EF} \right)^{S} \frac{a_{b-1,t-1}+H_{b-1,t-1}}{\eta_{b-1,t-1}} \tag{4.36}$$

and dividing this with (4.35) implies

$$\frac{Q_{b+1,t+1}}{Q_{b,t}} = \begin{cases} \left(\frac{1+r_{t+1}\left(1-t_{t+1}^r\right)}{1+\theta} \frac{P_t}{P_{t+1}} \right)^{S} & \text{for } 18 \leq b < 77 \\ \xi^{S} \left(\frac{1+r_{t+1}\left(1-t_{t+1}^r\right)}{1+\theta} \frac{P_t}{P_{t+1}} \right)^{S} & \text{for } b = 77 \end{cases} \tag{4.37}$$

which is the standard Keynes Ramsey consumption smoothing rule. It states that the increase in the utility index per adult-equivalent from one period to the next is a function of the following factors: First, the ratio of 1 plus the interest rate after tax, to 1 plus the pure rate of time preference. Second, the ratio of the value of this period's price index to the next period's. If the product of these ratios is larger than 1, then the utility index is increasing through time.

4.4 The optimal bequest decision

The final decision, that a given generation of households have to make is the bequest decision. The result of this decision is easily derived given the specification of the

[6] In the limit where S equals 1 - corresponding to the case of logarithmic utility function - the expression (4.35) reduces to $Q_{b,t} = \xi_b \frac{1}{1+\theta} \cdot \frac{(a_{b-1,t-1}+H_{b-1,t-1})[1+r_t(1-t_t^r)]}{P_t} N_{b-1,t-1}^{EF}$. So consumption is simply the real value of total wealth after payment of interest measured in todays prices multiplied by the one-period discount factor.

utility function (4.8). By the construction of this function, the bequest enters similar to consumption in an extra fictitious period after the final period of consumption ("period 78"). The weight attached to bequest, ξ, may however differ significantly from the weight 1, which is implicitly attached to instantaneous utility at a given age. In the simulations we assume that $\xi = 1$ and thus the weight representing the preferences for bequest, is similar to the weight representing preferences for an instantaneous utility in a given year. The construction implies that the optimal bequest can be found using the Keynes-Ramsey rule (4.37) adjusted by the preference parameter in the discounting

$$Q_{78,t} = \left(\xi \frac{1 + r_t \left(1 - t_t^r\right)}{1 + \theta} \frac{P_{t-1}}{P_t} \right)^S Q_{77,t-1}$$

4.4.1 Transforming bequest into inheritance

Inheritance is always transferred to the heirs 78 years after the birth of their mother, no matter whether she is still alive at the time or has passed away earlier. The inheritance is namely at all times distributed through the households and is paid out when the household ceases to exist. In the following the term "heir" is used about a person only if he or she actually receives inheritance in the given period of time. Therefore our definitions imply that heirs must have an age h between 29 and 60 - both numbers included. To find the distribution of heirs, we use the distribution of births as represented in the Ψ-matrix. First define the column vector, $\vec{\Psi}^{78}$, with 32 elements, containing for each of the age groups from 29 to 60 respectively the part of the group in question, whose mother was born 78 years ago. Using the notation introduced in the sub section "*Children*", $\vec{\Psi}^{78}$ can be written as below

$$\vec{\Psi}^{78} \equiv \begin{bmatrix} \psi_{29,78} \\ \psi_{30,78} \\ \vdots \\ \psi_{60,78} \end{bmatrix} = \begin{bmatrix} \psi_{0,49} \\ \psi_{0,48} \\ \vdots \\ \psi_{0,18} \end{bmatrix} \tag{4.38}$$

The equality is due to the earlier assumptions that the maternal distribution is time invariant and that the mortality rate of people is independent of the age of their mother. Define next the row vector, $\vec{N}_t^{29-60}$, with 32 elements, containing the total number of men and women grouped by age (first column is the number of persons between 29 and 30 years and the last contains the number of persons between 60 and 61 years). The number of heirs at time t, N_t^H, is then determined by the vector product

$$\vec{N}_t^{29-60} \times \vec{\Psi}^{78} = N_t^H \tag{4.39}$$

This enables us to find the inheritance per heir from the household that turns 78 years in period t. Let $B_{78,t}^{in}$ be this amount. $B_{78,t}^{in}$ is then defined as

$$B_{78,t}^{in} = \frac{P_t Q_{78,t} \cdot N_{78,t}^{EF}}{N_t^H} \tag{4.40}$$

where $P_t Q_{78,t}$ is the nominal bequest per adult-equivalent in the generation that turns 78 years in period t.

To distribute the inheritance to the households, it is necessary to divide the heirs of the household which turns 78 years into males and females, as the males have to be distributed according to the age of their wife. Let $N_{b,t}^{FH}$ denote the number of female heirs at the age of b years. $N_{b,t}^{FH}$ is defined as

$$N_{b,t}^{FH} = N_{b,t}^F \cdot \psi_{b,78} = N_{b,t}^F \cdot \psi_{0,78-b} \quad \text{for } b \in [29; 60] \tag{4.41}$$

The inheritance of women aged b in period t, $B_{b,t}^{inF}$, is then given by

$$B_{b,t}^{inF} = \frac{N_{b,t}^{FH} \cdot B_{78,t}^{in}}{N_{b,t}^F} \quad \text{for } b \in [29; 60] \tag{4.42}$$

$N_{b,t}^{FH} \cdot B_{78,t}^{in}$ is the total inheritance for women of age b. This is divided by the total number of women, $N_{b,t}^F$, in the age group in question. Another way to put it is, that the inheritance for a person, $B_{78,t}^{in}$, is received only by the part of the women, who is heirs - namely $\frac{N_{b,t}^{FH}}{N_{b,t}^F}$.

Similarly, for men we define the number of male heirs at the age of a years in period t, $N_{a,t}^{MH}$, as

$$N_{a,t}^{MH} = N_{a,t}^M \cdot \psi_{a,78} = N_{a,t}^M \cdot \psi_{0,78-a} \quad \text{for } a \in [29; 60] \tag{4.43}$$

The inheritance of men aged a in period t, $B_{a,t}^{inM}$, is then given by

$$B_{a,t}^{inM} = \frac{N_{a,t}^{MH} \cdot B_{78,t}^{in}}{N_{a,t}^M} \quad \text{for } a \in [29; 60] \tag{4.44}$$

Here we have used the assumption of the representative households. The number of men of age a in each household has to be large enough to ensure that the part of these who receives inheritance doesn't vary across households. So in all the generations of households the share of men at a given age (any age between 29 and 60) who are heirs is the same due to the law of large numbers[7].

Note that households of ages both greater than 60 and less than 29 can receive inheritance, since men in the ages from 29 to 60 can be part of these households. The inheritance of the female aged b years and the one of males aged a years in period t, (4.42) and (4.44) respectively, is used in the relation (4.3) which defines the total non-interest income of the household of age b years in period t.

4.5 Intratemporal optimization

Given that the index $Q_{b,t}$ and the labor supply $\ell^F_{b,t}$, $\ell^M_{a,b,t}$ for a, $b \in [18; 60]$ have been determined, the consumption index $C_{b,t}$ can be found from the relation

$$C_{b,t} = Q_{b,t} + Z_{b,t} \tag{4.45}$$

where $Z_{b,t}$, using the optimal labor choice $\ell^F_{b,t} = \ell^M_{a,b,t} = \ell_t$, is given by

$$Z_{b,t} = \begin{cases} f(\ell_t)\left(\frac{N^F_{b,t}}{N^{EF}_{b,t}} + \frac{1}{N^{EF}_{b,t}} \sum\limits_{a=18}^{60} N^M_{a,t} \cdot \omega_{a,b}\right) & \text{for} \quad 18 \leq b < 61 \\ f(\ell_t) \frac{1}{N^{EF}_{b,t}} \sum\limits_{a=18}^{60} N^M_{a,t} \cdot \omega_{a,b} & \text{for} \quad 61 \leq b < 78 \end{cases} \tag{4.46}$$

The total consumption per adult equivalent for the period in question, $C_{b,t}$, determined in (4.45) is given for the atemporal optimization problem of the households which is about distributing total household consumption per adult equivalent on its components.

Figure 13 illustrates the assumptions about the intratemporal sub utility function of households. At the top level private consumption goods which are supplied from the private sector substitutes private consumption goods supplied from the public sector. At the bottom level privately produced consumption goods of domestic and foreign origin substitute each other. There are no imports of the publicly produced good. This "utility tree" is specified as nested CES-functions. The derivation of the intratemporal

[7] This argument was not needed for the women, since all the women of a given age are connected to the same generation of household.

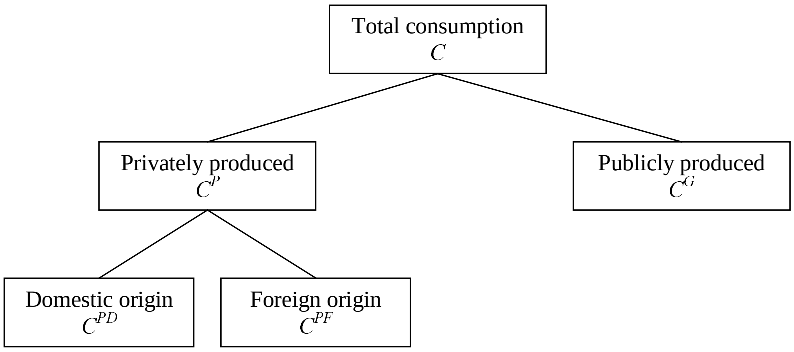

Figure 13 The intratemporal utility function of the household

consumer demand functions therefore proceeds analogously to the derivation of the intratemporal factor demand systems of the production sectors, cf. chapter 2.

Formally, the household chooses its demand for the different consumption goods in each period of time by minimizing the cost of obtaining the index $C_{b,t}$. Thus, at the top level demands are the solutions to the following minimization problem

$$\min_{C_{b,t}^P, C_{b,t}^G} \quad \left(1+t_t^{VAT}\right)\left(p_t^{CP}+t_t^{CP})C_{b,t}^P+(p_t^G+t_t^{CG})C_{b,t}^G\right) \tag{4.47}$$

$$s.t. \tag{4.48}$$

$$C_{b,t} = Q_{b,t} + Z_{b,t} \;, \quad \text{where}$$

$$C_{b,t} \equiv \left(\mu_{CP}\left(C_{b,t}^P\right)^{\frac{\sigma_C-1}{\sigma_C}} + \mu_{CG}\left(C_{b,t}^G\right)^{\frac{\sigma_C-1}{\sigma_C}}\right)^{\frac{\sigma_C}{\sigma_C-1}}$$

where the total consumption index $C_{b,t}$ is considered given, $C_{b,t}^P$ is the demand for the index of the private consumption good delivered from the private production sector (domestic and foreign), and $C_{b,t}^G$ is the demand for the private consumption good delivered from the domestic public sector (there are no imports of publicly produced consumption goods). $C_{b,t}^P$ and $C_{b,t}^G$ are defined per adult-equivalent of the household aged b years at time t. p_t^{CP} is the price index of privately produced consumption goods net of taxes, while p_t^G is the output price of the governmental production sector. t_t^{CP} and t_t^{CG} are quantity excise taxes on the two consumption goods. t_t^{VAT} is the VAT rate (representing all ad valorem taxes). $\sigma_C > 0$ (and $\sigma_C \neq 1$)is the elasticity of substitution between the two goods, and $\mu_{CP} > 0$ and $\mu_{CG} > 0$ are the weights attached to consumption of the two goods in the CES sub-utility function. Using Appendix B, theorem 1, we obtain the demand functions for the two goods

$$C_{b,t}^P = (\mu_{CP})^{\sigma_C}\left(\frac{\left(1+t_t^{VAT}\right)\left(p_t^{CP}+t_t^{CP}\right)}{P_t}\right)^{-\sigma_C} C_{b,t}$$

$$C^G_{b,t} = (\mu_{CG})^{\sigma_C} \left(\frac{\left(1 + t^{VAT}_t\right)\left(p^G_t + t^{CG}_t\right)}{P_t} \right)^{-\sigma_C} C_{b,t}$$

and the consumer price index is determined as

$$P_t = \left(1 + t^{VAT}_t\right) \left[(\mu_{CP})^{\sigma_C} \left(p^{CP}_t + t^{CP}_t\right)^{1-\sigma_C} + (\mu_{CG})^{\sigma_C} \left(p^G_t + t^{CG}_t\right)^{1-\sigma_C} \right]^{\frac{1}{1-\sigma_C}} \quad (4.49)$$

At the bottom level the demand for the index of the privately produced consumption good is splitted into the demands for the parts of domestic and foreign origin respectively by solving the following cost minimization problem

$$\min_{C^{PD}_{b,t}, C^{PF}_{b,t}} \quad p^P_t C^{PD}_{b,t} + C^{PF}_{b,t} \quad (4.50)$$

$$s.t. \quad (4.51)$$

$$C^P_{b,t} \equiv \left(\mu_{CPD} \left(C^{PD}_{b,t}\right)^{\frac{\sigma_{CP}-1}{\sigma_{CP}}} + \mu_{CPF} \left(C^{PF}_{b,t}\right)^{\frac{\sigma_{CP}-1}{\sigma_{CP}}} \right)^{\frac{\sigma_{CP}}{\sigma_{CP}-1}}$$

where the index $C^P_{b,t}$ is considered given, $C^{PD}_{b,t}$ is the demand for privately produced private consumption good of domestic origin and $C^{PF}_{b,t}$ is the demand for the privately produced private consumption good of foreign origin. The quantities are again measured per adult-equivalent of the household aged b years at time t. $\sigma_{CP} > 0$ (and $\sigma_{CP} \neq 1$) is the elasticity of substitution between the two goods, and $\mu_{CPD} > 0$ and $\mu_{CPF} > 0$ are the weights attached to consumption of the two goods in the CES subutility function. Again using appendix B, theorem 1, we obtain the demand functions for the two goods

$$C^{PD}_{b,t} = (\mu_{CPD})^{\sigma_{CP}} \left(\frac{p_t}{p^{CP}_t} \right)^{-\sigma_{CP}} C^P_{b,t} \quad (4.52)$$

$$C^{PF}_{b,t} = (\mu_{CPF})^{\sigma_{CP}} \left(\frac{1}{p^{CP}_t} \right)^{-\sigma_{CP}} C^P_{b,t} \quad (4.53)$$

and the price index of $C^P_{b,t}$ net of indirect taxes is determined as

$$p^{CP}_t = \left((\mu_{CPD})^{\sigma_{CP}} (p_t)^{(1-\sigma_{CP})} + (\mu_{CPF})^{\sigma_{CP}} \right)^{\frac{1}{1-\sigma_{CP}}} \quad (4.54)$$

The interpretation of this consumer demand system is analogous to the interpretation of the input demand equations of the firms, cf. chapter 2.2.1-2.2.2.

4.6 Very old persons

The planning horizon of the household expires when the woman who is the head of the household becomes 78 years old. The surviving individuals of such households are named very old persons although they include (a small number of) younger men married to a women at least 78 years old. It is assumed that the only economic activities of the very old persons is to recieve the ordinary average amount of social security pensions, pay the normal tax out of this amount and consume the remaining part, i.e. they do not save and accumulate new wealth. Further, it is assumed that their intratemporal utility function have the same functional form and the same specific values of its parameters as the intratemporal utility function of ordinary households. Therefore the very old persons allocate their total consumption expenditures on the different consumer goods in the same proportions as the ordinary households, and the CES-price indices of composite consumer goods are the same for very old persons and for ordinary households.

Formally, the income of each very old person, $y_{o,t}$,is defined as

$$y_{o,t} \equiv f_t^P - T_t^P \left(f_t^P\right) \tag{4.55}$$

where $f_t^P - T_t^P \left(f_t^P\right)$ as usual denotes the disposable social security pension. The very old persons obtain the demand for the different consumption goods in each period of time by maximizing the utility index $C_{o,t}$. Thus, at the top level demands per very old person are the solutions to the following maximization problem

$$\max_{C_{o,t}^P, C_{o,t}^G} \quad C_{o,t} \equiv \left(\mu_{CP}\left(C_{o,t}^P\right)^{\frac{\sigma_C-1}{\sigma_C}} + \mu_{CG}\left(C_{o,t}^G\right)^{\frac{\sigma_C-1}{\sigma_C}}\right)^{\frac{\sigma_C}{\sigma_C-1}} \tag{4.56}$$

$$s.t. \tag{4.57}$$

$$\left(1+t_t^{VAT}\right)\left(p_t^{CP}+t_t^{CP})C_{o,t}^P + (p_t^G + t_t^{CG})C_{o,t}^G\right) = y_{o,t}$$

where the total income, $y_{o,t}$ is considered given, $C_{o,t}^P$ is the demand for the index of the private consumption good delivered from the private production sector (domestic and foreign), and $C_{o,t}^G$ is the demand for the private consumption good delivered from the domestic public sector (there are no imports of publicly produced consumption goods). As ususal, p_t^{CP} is the price index of privately produced consumption goods net of taxes, p_t^G is the output price of the governmental production sector. t_t^{CP} and t_t^{CG} are quantity excise taxes on the two consumption goods, t_t^{VAT} is the VAT rate and $\sigma_C > 0$, $\mu_{CP} > 0$ and $\mu_{CG} > 0$ are parameters of the CES utility function being identical to the CES utility function of ordinary households stated in (4.47). Using Appendix B, section 1.2, we obtain the demand functions for the two goods

$$C^P_{o,t} = (\mu_{CP})^{\sigma_C} \left(\frac{\left(1+t^{VAT}_t\right)\left(p^{CP}_t + t^{CP}_t\right)}{P_t} \right)^{-\sigma_C} C_{o,t} \tag{4.58}$$

$$C^G_{o,t} = (\mu_{CG})^{\sigma_C} \left(\frac{\left(1+t^{VAT}_t\right)\left(p^{G}_t + t^{CG}_t\right)}{P_t} \right)^{-\sigma_C} C_{o,t} \tag{4.59}$$

and the consumer price index is determined as

$$P_t = \left(1+t^{VAT}_t\right)\left[(\mu_{CP})^{\sigma_C}\left(p^{CP}_t + t^{CP}_t\right)^{1-\sigma_C} + (\mu_{CG})^{\sigma_C}\left(p^{G}_t + t^{CG}_t\right)^{1-\sigma_C}\right]^{\frac{1}{1-\sigma_C}} \tag{4.60}$$

where

$$P_t C_{o,t} = y_{o,t}$$

i.e. the CES utility index $C_{o,t}$ can be interpreted as a quantity index in $C^P_{o,t}$, $C^G_{o,t}$. The CES consumer price index (4.60) of the very old persons is identical to the CES consumer price index of (4.49) of the ordinary households due to the identical intratemporal utility functions.

At the bottom level the demand for the index of the privately produced consumption good, $C_{o,t}$, is splitted into the demands for the parts of domestic and foreign origin respectively by solving the following cost minimization problem

$$\min_{C^{PD}_{o,t}, C^{PF}_{o,t}} \quad p^P_t C^{PD}_{o,t} + C^{PF}_{o,t} \tag{4.61}$$

$$s.t. \tag{4.62}$$

$$C^P_{o,t} \equiv \left(\mu_{CPD}\left(C^{PD}_{o,t}\right)^{\frac{\sigma_{CP}-1}{\sigma_{CP}}} + \mu_{CPF}\left(C^{PF}_{o,t}\right)^{\frac{\sigma_{CP}-1}{\sigma_{CP}}} \right)^{\frac{\sigma_{CP}}{\sigma_{CP}-1}}$$

where the index $C^P_{o,t}$ is considered given, $C^{PD}_{b,t}$ is the demand for privately produced private consumption good of domestic origin, $C^{PF}_{o,t}$ is the demand for the privately produced private consumption good of foreign origin and σ_{CP}, μ_{CPD} and μ_{CPF} are the parameters of the CES sub-utility function again being identical to the corresponding sub-utility function (4.51) of ordinary households. Using appendix B, theorem 1, we obtain the demand functions for the two goods

$$C^{PD}_{o,t} = (\mu_{CPD})^{\sigma_{CP}} \left(\frac{p_t}{p^{CP}_t} \right)^{-\sigma_{CP}} C^P_{o,t} \tag{4.63}$$

$$C_{o,t}^{PF} = (\mu_{CPF})^{\sigma_{CP}} \left(\frac{1}{p_t^{CP}}\right)^{-\sigma_{CP}} C_{o,t}^{P} \tag{4.64}$$

and the price index of $C_{o,t}^{P}$ net of indirect taxes is determined as

$$p_t^{CP} = \left((\mu_{CPD})^{\sigma_{CP}} (p_t)^{(1-\sigma_{CP})} + (\mu_{CPF})^{\sigma_{CP}}\right)^{\frac{1}{1-\sigma_{CP}}} \tag{4.65}$$

which is identical to the corresponding price index (4.54) for the ordinary households.

This finishes the behavioral description of the households. In the next chapter we complete the description of the model by looking at the macroeconomic relations.

Chapter 5
MACROECONOMIC RELATIONS

In the previous three chapters, we have described the behavior of the private agents in the economy. This chapter defines the aggregate variables, describes the behavior of the foreign sector and the determination of the public budget and specifies the equilibrium conditions for the goods and the labor market.

Aggregation across generations involves some weighed sum formulas which are stated explicitly in separate formulas. Aggregation across producers is done by simple summations which are only stated where needed.

5.1 Aggregation across generations

The total aggregate demand for goods and supply of labor is found by adding together the demand and supply functions of the representative households of all generations alive, at a given moment in time. This is done for all periods of time t. For each generation of households, the level of consumption per adult-equivalent is weighted by the number of adult-equivalents in the generation of households in question. The consumption of the very old persons (who have survived the planning horizon of the household) must be added to obtain total consumption. The same procedure is applied with respect to the labor supply, except for the fact that in this case the labor supply per adult of a generation of households is weighted by the number of adults as children are assumed not to supply labor, and that the labor supply of the very old persons is zero.

The total number of very old persons is defined as

$$N_t^O = \sum_{b=78}^{\tilde{B}} N_{b,t}^F + \sum_{a=78}^{\tilde{A}} N_{a,t}^M - \sum_{b=18}^{77} \sum_{a=78}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} \tag{5.1}$$

where $\tilde{B}$ is the maximum age of a female, such that $\sum_{b=78}^{\tilde{B}} N_{b,t}^{F}$ is the sum of females who are older than 77 years at period t. Similarly $\tilde{A}$ is the maximum age of a male, such that $\sum_{a=78}^{\tilde{A}} N_{a,t}^{M}$ is the sum of males who are older than 77 years at period t. However, some of these old males are married to women, who are younger than 77 years. The behavior of these men is therefore already taken into consideration in the decision making of the households. Thus these men have to be subtracted from the number of very old people. The number of men older than 78 years married to women younger than 77 years is given by the sum $\sum_{b=18}^{77} \sum\limits_{a=78}^{\tilde{A}} N_{a,t}^{M} \cdot \omega_{a,b}$.

The level of aggregate demand and supply of the different goods are defined below as

$$C_t^P = \sum_{b=18}^{77} N_{b,t}^{EF} C_{b,t}^P + N_t^O C_{o,t}^P \quad \text{for all } t \tag{5.2}$$

where C_t^P is the total domestic consumption demand for the privately produced good

$$C_t^G = \sum_{b=18}^{77} N_{b,t}^{EF} C_{b,t}^G + N_t^O C_{o,t}^G \quad \text{for all } t \tag{5.3}$$

where C_t^G is the total domestic consumption demand for the governmentally produced good

$$C_t^{PD} = \sum_{b=18}^{77} N_{b,t}^{EF} C_{b,t}^{PD} + N_t^O C_{o,t}^{PD} \quad \text{for all } t \tag{5.4}$$

where C_t^{PD} is the total domestic consumption demand for the privately produced domestic good

$$C_t^{PF} = \sum_{b=18}^{77} N_{b,t}^{EF} C_{b,t}^{PF} + N_t^O C_{o,t}^{PF} \quad \text{for all } t \tag{5.5}$$

where C_t^{PF} is the total domestic consumption demand for the privately produced foreign good

$$\begin{aligned} L_t^s &= \sum_{b=18}^{60} N_{b,t}^F \ell_t + \sum_{b=18}^{77} \sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \cdot \ell_t && (5.6) \\ &= \ell_t \left(\sum_{b=18}^{60} N_{b,t}^F + \sum_{b=18}^{77} \sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \right) \\ &= \ell_t N_t^W \quad \text{for all } t && (5.7) \end{aligned}$$

where L_t^s is the total labor supply measured in hours and

$$N_t^W = \left(\sum_{b=18}^{60} N_{b,t}^F + \sum_{b=18}^{77} \sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \right) \tag{5.8}$$

is the number of persons in the work force.

$$A_t = \sum_{b=18}^{77} N_{b,t}^{EF} a_{b,t} \quad \text{for all } t \tag{5.9}$$

where A_t is the total stock of non-human wealth held by domestic consumers

$$TR_t = \sum_{b=18}^{77} \left(N_{b,t}^F \left(TR_{b,t}^F + TRT_{b,t}^F \right) + \sum_{a=18}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} \cdot \left(TR_{a,t}^M + TRT_{a,t}^M \right) \right) \quad \text{for all } t \tag{5.10}$$

where TR_t is the sum of age specific transfers

$$\begin{aligned} UB_t &= b_t \left(\sum_{b=18}^{77} \sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} \cdot \left(\bar{\ell} - \ell_t \right) + \sum_{b=18}^{60} N_{b,t}^F \left(\bar{\ell} - \ell_t \right) \right) \\ &= b_t \left(\bar{\ell} - \ell_t \right) \left(\sum_{b=18}^{77} \sum_{a=18}^{60} N_{a,t}^M \cdot \omega_{a,b} + \sum_{b=18}^{60} N_{b,t}^F \right) \\ &= b_t \left(\bar{\ell} - \ell_t \right) N_t^W \quad \text{for all } t \end{aligned} \tag{5.11}$$

where UB_t is total gross expenditures on unemployment benefits

$$FP_t = f_t^P \left(\sum_{b=18}^{77} \sum_{a=61}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} + \sum_{b=61}^{77} N_{b,t}^F + N_t^O \right) \quad \text{for all } t \tag{5.12}$$

where FP_t is the total gross expenditure on social security pensions

$$TW_t = \sum_{b=18}^{77} N_{b,t}^{AF} \tau_t^W \quad \text{for all } t \tag{5.13}$$

where TW_t is the total transfers to households from abroad

$$\Upsilon_t = \sum_{b=18}^{77} N_{b,t}^{AF} \tau_t \quad \text{for all } t \tag{5.14}$$

where Υ_t is the total value of the lump sum transfer to the consumers.

5.2 The foreign sector

The domestic economy is integrated in the world economy, through trade and capital flows. As to financial capital, we assume that domestic corporate and government bonds are perfect substitutes for foreign bonds in a perfect world bond market. Residence based taxation of interest income, implies that the domestic pre-tax interest rate is equal to the foreign pre-tax interest rate, which is assumed to be fixed (through time) in units of the foreign good.

The accumulation of domestic claims on the rest of the world, is determined by the current account

$$F_t = (1 + r_t) F_{t-1} + TW_t + TWG_t + \sum_{i=P,G} p_t^i \left(1 + t_t^{iX}\right) X_t^i \tag{5.15}$$

$$-\frac{1}{1 + t_t^t} \left(C_t^{PF} + \sum_{i=P,G} \left(M_t^{iPF} + I_t^{iF} \right) \right) \tag{5.16}$$

where F_t is the domestic net stock of foreign assets, TWG_t is total transfers to the domestic government from abroad, X_t^i is exports from production sector i, t_t^{iX} is the associated export tax (normally negative, i.e. an export subsidy). Total import consists of imported consumer goods, C_t^{PF}, imported materials to the production sectors, M_t^{iPF}, and imported investment goods to the production sectors, I_t^{iF}. t_t^t is the tariff. For convenience we assume that an identical tariff applies to all types of goods. Since by definition the foreign price inclusive tariff is equal to 1, then foreign price net of tarif must be $\frac{1}{1+t_t^t}$.

Alternatively, the stock of foreign assets may be expressed as the difference between the stock of total private non-human wealth, A_t on the one hand and sum of the value of assets issued by domestic firms and the public debt on the other hand

$$F_t = A_t - (V_t + B_t^c + B_t^g) \tag{5.17}$$

where V_t is the value of shares in domestic firms, B_t^c is domestic corporate debt, and B_t^g is domestic public bonds.

(5.17) defines the balance between all the households assets. For most variables the model is only solved for $t \geq 1$. However for some variables including the value of the firm, the model is also solved for $t = 0$.[1] It is therefore likely that the value of the firm at the end of period 0, V_0, changes compared with the calibrated value at the

[1] See chapter 7.7 on the solution of the model for an elaboration of this point.

same point in time, V_{t_0}. It is assumed that no other item of households' assets change at the end of period 0, i.e. according to (5.17) the sum of households' assets, A_0, must change by the amount $V_0 - V_{t_0}$. Further, it is assumed that the asset holdings of all generations change by an equal relative amount. The relation between the model solution value of generation b's assets at the end of period 0, $a_{b,0}$, and the calibrated (initial) value for the same measure, a_{b,t_0}, can therefore be determined using (5.9) as

$$a_{b,0} = a_{b,t_0} \left(1 + \frac{V_0 - V_{t_0}}{\sum_{b=18}^{77} N_{b,t}^{EF} a_{b,t_0}} \right)$$

Although the economy is small in the sense that it does not affect the world market interest rate, we assume that the world production of the good produced in the domestic country **is affected** by the level of production in the domestic country. Thus in this respect the country is not small in its own output market. This implies that the economy has endogenous terms of trade with the rest of the world.

Only the the private sector supplies exports to the world market. The foreign demand for the privately produced domestic good, can be thought of as demand functions derived from intertemporal optimization of foreigners. For simplicity it is assumed that the foreign demand functions for the domestic goods are isoelastic, and that the positions of the demand curves are fixed through time

$$X_t^i = \chi^i \left(\left(1 + t_t^{iX}\right) p_t^i \right)^{-\varepsilon^i} \quad i = P, G \tag{5.18}$$

where $\chi^i > 0$ are constants, $\varepsilon^i > 0$ is the price elasticity in the export demand.

As is well known from classical trade theory, the introduction of endogenous terms of trade implies that there on welfare grounds exists a positive optimal tariff. As no tariffs are introduced in the present version of the model, this implies that any policy that introduces a marginal cut back on domestic production, and therefore a positive terms of trade effect will *ceteris paribus* tend to increase the utility of the representative consumer in the domestic economy at the expense of consumers abroad. Thus the endogenous terms of trade introduces the possibility of using "beggar thy neighbor policies" to increase domestic welfare. As will be apparent in chapter 8, this has significant effects on most of the policy experiments.

5.3 The consumption and the budget of the public sector

Besides being a producer, the (main) role of the government is to collect taxes, to finance unemployment and pension benefits, other personal transfers such as family allowance transfers, and public consumption which is government purchases of goods and services supplied solely from the governmental production sector. Notice that this public consumption does not explicitly enter the utility function of the agents. One interpretation of this is, that the utility function of the agents has an implicit additively separable argument containing the public consumption, thus the amount of public consumption does not affect the marginal utility of private consumption and vice versa. This specification of government consumption in the utility function (i.e. it is omitted) implies that the model is ill-suited to analyze welfare effects of changes in the size of government consumption *per se*[2].

The government faces an intertemporal budget constraint (the so-called No Ponzi Game Condition), which implies that the discounted value of government debt has to converge to zero for time approaching infinity. This means, that in the long run the increase in the public debt must be less than the interest payments on the debt.

The consolidated budget constraint of the government is derived by using the savings identity and the No Ponzi Game condition. The procedure is similar to the derivation of the consolidated budget constraint of the household in chapter 4. This yields the result that the present discounted value of the stream of tax revenues must be greater than or equal to the sum of the public debt and the present discounted value of the stream of government expenditures

$$\sum_{s=t}^{\infty} \frac{\widehat{tax}_s}{\prod_{v=t+1}^{s}(1+r_t)} = \sum_{s=t}^{\infty} \frac{\hat{G}_s}{\prod_{v=t+1}^{s}(1+r_t)} + B_t^g \tag{5.19}$$

where $\widehat{tax}_s$ is the total tax revenue at time s, $\hat{G}_s$ is the total public expenditures net of interest payments at time s, B_t^g is the stock of outstanding public debt at time t.

Violating this means, that private agents are not willing to hold government bonds. Consistency thus requires, that a policy reaction function is assumed to fulfill the intertemporal budget constraint.

A simple policy reaction rule, is to assume that the government uses a lump sum transfer/tax to keep the government budget balanced in every period, i.e. to keep the government debt fixed. This has the convenient feature, that no dynamic effects

[2] This would yield a rather peculiar policy finding, since utility is inversely correlated with the size of government purchases - therefore diminishing government expenditures will unambiguously increase welfare! This problem can be solved by altering the utility function to take government expenditures into account. However, this method is not used here, since it would give cause to an even larger problem when calibrating the model; how to determine the weight to place on the expenditures of the public sector?

are fed into the economy from a change in the public debt. However, the assumption implies that reducing any distortionary tax rate without specifying a source of finance, tends to lead to a welfare improvement since this amounts to replacing a distortionary tax with a non distortionary. Thus welfare implications of underfinanced reforms tends be upward biased. The opposite is true for overfinanced reforms. Keeping these biases in mind, we assume the simple (and unrealistic) policy rule. The government budget constraint then becomes

$$\begin{aligned}\Upsilon_t &= -r_t B_{t-1}^g - \hat{g}\hat{Y}_t - p_t^{GI} I_t^G - TR_t - UB_t - FP_t - FO_t \\ &\quad + TWG_t + tax_t + p_t^G MPK_t^G K_{t-1}^G \end{aligned} \tag{5.20}$$

where $\hat{g}\hat{Y}_t$ is the public consumption. $0 < \hat{g} < 1$ is a constant fraction and

$$\hat{Y}_t = \sum_{i=P,G} p_t^i Y_t^i - p_t^{iM} M_t^i \tag{5.21}$$

is nominal GDP. $p_t^G MPK_t^G K_{t-1}^G$ is the total reward to the governmental capital stock. tax_t is the tax revenue from distortionary taxes in the economy. We define the tax revenues from each distortionary tax separately (and valid for all t)

$$
\begin{aligned}
LT_t &= T_t^w \left(W_t\right) \ell_t N_t^W && (5.22)\\
BT_t &= T_t^b \left(b_t\right) \left(\bar{\ell} - \ell_t\right) N_t^W && (5.23)\\
TRT_t &= \sum_{b=18}^{77} N_{b,t}^F T_{b,t}^T \left(TRT_{b,t}^F\right) + \sum_{b=18}^{77} \sum_{a=18}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} \cdot T_{a,t}^T \left(TRT_{a,t}^M\right) && (5.24)\\
PT_t &= T_t^P \left(f_t^P\right) \left(\sum_{b=61}^{77} N_{b,t}^F + \sum_{b=18}^{77} \sum_{a=61}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b} \right) && (5.25)\\
ET_t &= t^a W_t \ell_t N_t^W && (5.26)\\
VAT_t &= t^{VAT} \left(\left(p_t^{CP} + t_t^{CP}\right) C_t^P + \left(p_t^G + t_t^{CG}\right) C_t^G \right) && (5.27)\\
RC_t &= t_t^{CP} C_t^P + t_t^{CG} C_t^G && (5.28)\\
RT_t &= t_t^r r_t \left(A_{t-1} - V_{t-1}\right) && (5.29)\\
DT_t &= t_t^d D_t && (5.30)\\
CT_t &= t_t^c \left[p_t^P Y_t^P \text{-} \left(1 + t_t^a\right) W_t \ell_t N_t^W \text{-} p_t^{PM} M_t^P \text{-} \hat{\delta} \hat{K}_{t-1}^P \text{-} r_t g p_{t-1}^{PI} K_{t-1}^P \right] && (5.31)\\
GT_t &= \begin{cases} t_t^g \left(V_t - V_{t-1}\right) + t_0^g \left(V_0 - V_{t_0}\right) & for\ t = 1 \\ t_t^g \left(V_t - V_{t-1}\right) & for\ t > 1 \end{cases} && (5.32)\\
MT_t &= \sum_{i=P,G} t_t^{iM} \left(p_t^P M_t^{iPD} + M_t^{iPF} + p_t^G M_t^{iG}\right) && (5.33)\\
IT_t &= \sum_{i=P,G} t_t^{iI} \left(p_t^P I_t^{iD} + I_t^{iF}\right) && (5.34)\\
XT_t &= \sum_{i=P,G} t_t^{iX} p_t^i X_t^i && (5.35)\\
TT_t &= t_t^t \frac{1}{1 + t_t^t} \left[C_t^{PF} + \sum_{i=P,G} \left(M_t^{iPF} + I_t^{iF}\right) \right] && (5.36)
\end{aligned}
$$

where LT_t is the tax-revenue from labor income tax on employed, BT_t is the tax-revenue from labor income tax on unemployed. These revenues are differentiated as employed persons are subject to the labor market contribution tax whereas unemployed are not. TRT_t is the tax-revenue from taxation of age specific transfers, PT_t is the tax-revenue from taxation of public pensions, ET_t is the tax-revenue from the payroll tax, VAT_t is the tax revenue from value added consumption taxes, RC_t is the tax revenue from consumption quantity taxes, RT_t is the tax-revenue from the tax on interest payments, DT_t is the tax-revenue from the taxation of dividends, CT_t is the tax-revenue from corporate taxation, GT_t is the tax-revenue from the capital gains tax, MT_t is the revenue from the tax on materials, IT_t is the revenue from the tax on investment, XT_t is the negative revenue from the export subsidies, and finally, TT_t is the domestic tariff-revenue.

The capital gains tax revenue, GT_t, is simply calculated as the tax rate, t_t^g, times the increase in the value of the firm, $V_t - V_{t-1}$, during period t for all periods except the first. For the first period we add a capital gains tax for period 0, if the value of the firm at the end of period 0, V_0, has changed compared with the calibrated value at the same point in time, V_{t_0}. This is because the model is not solved for the tax revenue (and most other variables) in period 0.[3]

The total tax revenue is defined as

$$\begin{aligned} tax_t &= LT_t + BT_t + TRT_t + PT_t + ET_t + VAT_t + RC_t + RT_t + DT_t + \\ &\quad CT_t + GT_t + MT_t + IT_t + XT_t + TT_t \end{aligned} \tag{5.37}$$

5.4 Equilibrium conditions

To close the model, we need the market clearing conditions for the domestic goods markets and the market for labor. These are given by

$$Y_t^P = \sum_{i=P,G} \left(M_t^{iPD} + I_t^{iD} \right) + C_t^{PD} + X_t^P \tag{5.38}$$

$$Y_t^G = \sum_{i=P,G} M_t^{iG} + C_t^G + \frac{\hat{g}\hat{Y}}{p_t^G} + X_t^G \tag{5.39}$$

$$L_t^s = L_t^P + L_t^G \tag{5.40}$$

[3]See chapter 7.7 on the solution of the model for an elaboration of this point.

Chapter 6
STATIONARY STATE

The purpose of this chapter is twofold. First, we demonstrate that the model has a unique stationary state for the current calibration. It is quite important to be sure that there is only one stationary state in the model. In the occurrence of multiple stationary states, instability for some of the stationary states will mostly be the case. There is therefore a possibility for unintentionally calibration of a unstable stationary state, which is undesirable. Second, we describe the behavior of the households of different ages in the stationary state of the model. The desciption is based on numerical simulations, i.e. the predictions of the model given the macroeconomic calibration.

Uniqueness of the stationary state is demonstrated partly analytically, partly numerically. First it is shown mathematically that in stationary state there exists an aggregate version of the model. In this aggregate model it is possible to derive an aggregate supply curve. As no non-produced good exists, there is no nominal anchor and therefore no aggregate demand curve in the standard definition. However, we derive the aggregate demand as a function of the supply in the economy by ignoring the market clearing condition in the goods market. Using the "pseudo demand relationship" and the aggregate supply curve we demonstrate numerically that there is a unique stationary equilibrium of the current calibration.

6.1 The supply side of the economy

A stationary state equilibrium is defined as a temporary equilibrium, where the economic variables are constant through time. To analyze the stationary state we repeat the first order conditions from the general case ignoring the time index.

The first part of the analysis concerns the aggregate supply schedule. The definition of the aggregate supply is: The supply of goods from the firms as a function of the price of the domestic product given that the labor market is in equilibrium and that the firms are not rationed in their demands for imported inputs.

The procedure to derive the aggregate supply is: First, observe that the system of first order conditions defines the demand for a specific input as a function of the two remaining inputs plus nominal wage and the output price. Second, use the homogeneity of degree 0 of the marginal products to determine inputs relative to the labor input. The system of first order conditions now determines the input of material and capital relative to the input of labor and determines a relationship between the nominal wage and the output price which has to be fulfilled in optimum. The final step is insert the equilibrium in the labor market, which implies that demand for labor is equal to supply of labor, where the latter is determined by the relation between the nominal wage and the output price which was a result of the system of first order condition of the firm.

In the following we perform these 3 steps to derive the aggregate supply curve:

6.1.1 Step 1: The system of first order conditions of the firm

The first order conditions of the firms are given in the equations (2.17) - (2.21) at page 26. In the stationary state they become

$$\frac{\partial F}{\partial L}(M,K,L) = (1+t^a)\frac{W}{p} \tag{6.1}$$

$$\frac{\partial F}{\partial M}(M,K,L) = \frac{p^M}{p} \tag{6.2}$$

$$\frac{1-t^d}{1-t^g}\left(1-g+(1-t^c)\frac{\partial \Phi}{\partial I}(I,K)\right) = \frac{\lambda_1}{p^I}+\lambda_2 \tag{6.3}$$

$$\tfrac{\partial F}{\partial K}(M,K,L) - \tfrac{\partial \Phi}{\partial K}(I,K) = \frac{1}{1-t^c}\left[\left[(1-t^c)\,r+\delta\right]g\frac{p^I}{p} + \tfrac{1-t^g}{1-t^d}\left(\tfrac{1-t^r}{1-t^g}r+\delta\right)\frac{\lambda_1}{p}\right] \tag{6.4}$$

$$\lambda_2 = \frac{\left(1-t^d\right)}{\left(1-t^g\right)}\frac{t^c\hat{\delta}}{\frac{(1-t^r)}{(1-t^g)}r+\hat{\delta}} \tag{6.5}$$

The interpretation of the equations (6.1) and (6.2) is similar to the one given in chapter 2. Relation (6.5) is a special case of relation (2.25) from page 28, where we have used the fact that the tax rate and the rate of interest is constant through time in the stationary state. Therefore λ_2 is the tax-adjusted discounted stream of the depreciation allowance in the entire time horizon. The discounting contain both the depreciationg in the book value of the firm and the tax-adjusted interest rate.

The first order conditions concerning the capital stock reduces even further compared to chapter 2. This follows from the fact that in the stationary state, the stock of capital is constant. Therefore net-investments are 0, and gross investments are equal to the depreciation of existing capital. Thus the equation for the accumulation of capital, (2.6) at page 22, becomes

$$I = \delta K \tag{6.6}$$

Substituting (6.5) and (6.6) into (6.3) and using that $\frac{\partial \Phi}{\partial I}$ is homogenous of degree 0, yields

$$\frac{\lambda_1}{p^I} = \frac{1-t^d}{1-t^g}\left(1-g+\left(1-t^c\right)\frac{\partial \Phi}{\partial I}(\delta,1) - \frac{\left(1-t^g\right)t^c\hat{\delta}}{\left(1-t^r\right)r+\left(1-t^g\right)\hat{\delta}}\right) \tag{6.7}$$

In the stationary state the shadow price of capital divided by the replacement cost of capital, $\frac{\lambda_1}{p^I}$, is given by the values of the capital income tax rates, the costs of installation of capital and the degree of debt financing in the firm.

Tobin's marginal q is defined as the ratio of the shadow price of capital to the replacement cost of capital (see Hayashi (1982)). In standard textbook macroeconomic models (without capital income taxation) this ratio is equal to 1 in the stationary state. Due to the fact that the cost of installation of capital is defined as a function of gross investment (as opposed to net-investments), this is not the case in the present model. The inclusion of the debt ratio of the firm in the expression above is due to the definition of λ_1 as the shadow price of capital to the owners of the stock of shares in the firm and not to the firm as such. As is clear from the expression (6.7) above the introduction of tax deductable depreciation rates implies that Tobin's q deviates from 1.

Substituting Tobin's q into (6.4) and using that $\frac{\partial \Phi}{\partial K}$ is homogenous of degree 0 implies that also the marginal product of capital in the stationary state is given as a function of the producer price, p and exogenous variables

$$\frac{\partial F}{\partial K}\left(M,K,L\right) = \beta^{ss}\left(p\right) \tag{6.8}$$

where

$$\begin{aligned}\beta^{ss}(p) &= \frac{p^I(p,1)}{p}\left[r\left(g+\tfrac{(1-t^r)}{(1-t^g)(1-t^c)}(1-g)\right)+\delta+\tfrac{(1-t^r)rt^c\left(\delta-\hat{\delta}\right)}{(1-t^c)\left[(1-t^r)r+(1-t^g)\hat{\delta}\right]}\right]+\\ &\quad \frac{\partial\Phi}{\partial K}(\delta,1)+\left(\frac{1-t^r}{1-t^g}r+\delta\right)\frac{p^I(p,1)}{p}\frac{\partial\Phi}{\partial I}(\delta,1)\end{aligned} \tag{6.9}$$

In standard textbook models of a small, open economy the marginal product of capital in the stationary state is given by the sum of the world interest rate and the rate of depreciation of capital. Ignoring taxes and the fact that the cost of installation of capital is a function of gross investment, we see that this is also the case in the present model.

Defining the cost of capital, ς, as the required return to a marginal unit of capital net of physical depreciation measured in units of output prices in the stationary state, we have that the marginal condition may be written as

$$\varsigma=\frac{\partial F}{\partial K}(M,K,L)-\delta\frac{p^I(p,1)}{p} \tag{6.10}$$

Using the conditions (6.8) and (6.9) yields the following expression for the cost of capital

$$\begin{aligned}\varsigma &= \frac{p^I(p,1)}{p}\left[r\left(g+\tfrac{(1-t^r)}{(1-t^g)(1-t^c)}(1-g)\right)+\tfrac{(1-t^r)rt^c\left(\delta-\hat{\delta}\right)}{(1-t^c)\left((1-t^r)r+(1-t^g)\hat{\delta}\right)}\right]+\\ &\quad \frac{\partial\Phi}{\partial K}(\delta,1)+\left(\frac{1-t^r}{1-t^g}r+\delta\right)\frac{p^I(p,1)}{p}\frac{\partial\Phi}{\partial I}(\delta,1)\end{aligned} \tag{6.11}$$

Observe that the cost of capital fals into three parts: cost of financing, ς^r, cost of depreciation, ς^δ, and cost of installation of capital, ς^Φ. The cost of financing is defined as

$$\varsigma^r=\frac{p^I(p,1)}{p}r\left(g+\frac{(1-t^r)}{(1-t^g)(1-t^c)}(1-g)\right) \tag{6.12}$$

If the tax system does not affect then the financing cost reduces to $\frac{p^I(p,1)}{p}r$. Observe that this is the case if either $(1-t^r)=(1-t^g)(1-t^c)$ such that the taxation of detained dividends is identical to the taxation of interest income or if the firm i 100% debt financed $(g=1)$. Observe also that the dividend tax rate, t^d, does not affect the financing costs. This is due to the assumption that the fraction $(1-g)$ of the investment i financed completely be detained earnings.

The cost of depreciation is defined as

$$\varsigma^\delta=\frac{p^I(p,1)}{p}\frac{(1-t^r)rt^c\left(\delta-\hat{\delta}\right)}{(1-t^c)\left((1-t^r)r+(1-t^g)\hat{\delta}\right)} \tag{6.13}$$

These costs are zero, if the rate of depreciation allowed by the tax system is identical to the rate of physical depreciation. In the Danish economy (and in the current calibration) the depreciation allowed by the tax system is higher than the physical depreciation, such that these costs are in fact negative.

The final part of the cost of capital is the cost of installation given as

$$\varsigma^{\Phi} = -\iota\left(\delta\right)^{\iota+1} + \left(\frac{1-t^{r}}{1-t^{g}}r+\delta\right)\frac{p^{I}\left(p,1\right)}{p}\left(1+\iota\right)\left(\delta\right)^{\iota} \tag{6.14}$$

where we have inserted the value of the partial derivatives of the $\Phi-$function in the stationary state

After these manipulations the system of first order conditions is reduced to (6.1), (6.2), and (6.8).

6.1.2 Step 2: Determining relative inputs and relative prices

In this section we define the capital-labor- and materials-labor-ratios k, and m respectively

$$k \equiv \frac{K}{L} \tag{6.15}$$

$$m \equiv \frac{M}{L} \tag{6.16}$$

As $F(M,K,L)$ is homogeneous of degree 1, the partial derivatives of this function are homogeneous of degree 0, which implies that the reduced system of first order conditions in the stationary state (6.1), (6.2), and (6.8) may be written as

$$\frac{\partial F}{\partial L}\left(m,k,1\right) = \left(1+t^{a}\right)\frac{W}{p} \tag{6.17}$$

$$\frac{\partial F}{\partial M}\left(m,k,1\right) = \frac{p^{M}\left(p,1\right)}{p} \tag{6.18}$$

$$\frac{\partial F}{\partial K}\left(m,k,1\right) = \beta^{ss}\left(p\right) \tag{6.19}$$

This has a non-analytical solution for a given price p

$$\begin{array}{rl} m = m\left(p\right), & m_{p}^{\prime} > 0 \\ k = k\left(p\right), & k_{p}^{\prime} > 0 \\ W = W\left(p\right), & W_{p}^{\prime} > 0 \end{array} \tag{6.20}$$

The signs of the first derivatives are derived in Appendix C. Observe that, m, k and W all increasing in the level of output price. As shown in Appendix C, this is also true for the product real wage, $\frac{W(p)}{p}$.

6.1.3 *Step 3: The equilibrium in the labor market*

In the stationary state the supply of labor determined from (4.26) is given as

$$\ell = \left(\frac{[W - T^w(W)] - \left[b - T^b(b)\right]}{\gamma_1 P} \right)^{\gamma} \tag{6.21}$$

and the aggregate supply of labor thereby

$$L = \ell N^W \tag{6.22}$$

Using the fact that the level of benefits is indexed by the wage rate as given by the indexation rule, (4.5) at page 60, we find that the numerator of (6.21) is an increasing function of the nominal wage. From the definition of the price index, P, given in equation (**??**) at page **??**, we have that the price index is an increasing function of the domestic output price. Finally inserting the relation between the nominal wage and the output price which is given by the reduced system of first order conditions in step 2 (see equation (6.20)) into the labor supply condition, implies that we may write this as

$$L = L(p) \equiv \left(\frac{[W(p) - T^w(W(p))] - \left[\varphi^b W(p)\left(1-t^\ell\right) - T^b\left(\varphi^b W(p)\left(1-t^\ell\right)\right)\right]}{\gamma_1 \frac{1}{2}(1+t^{VAT})\left((1-\mu_c)^E + (\mu_c)^E p^{1-E}\right)^{\frac{1}{1-E}}} \right)^{\gamma} N^W, \; L_p' > 0 \tag{6.23}$$

The aggregate supply of labor is increasing in the domestic output price. This is shown in Appendix C. The fact that labor supply (and thus employment) is increasing in the domestic output price implies that inputs of all factors are increasing in the domestic output price level. This follows from the fact that

$$M = M(p) \equiv m(p) \cdot L(p) \tag{6.24}$$

$$K = K(p) \equiv k(p) \cdot L(p) \tag{6.25}$$

Finally the aggregate supply function is deduced from (2.8) - see page 23 - and (6.6)

$$Y = Y^S(p) \equiv F[M(p), K(p), L(p)] - \Phi(\delta, 1) K(p) \tag{6.26}$$

This is the aggregate supply function. Obviously

$$Y_p^{S'} > 0$$

The aggregate supply curve based on the numerical solution is shown in figure 14

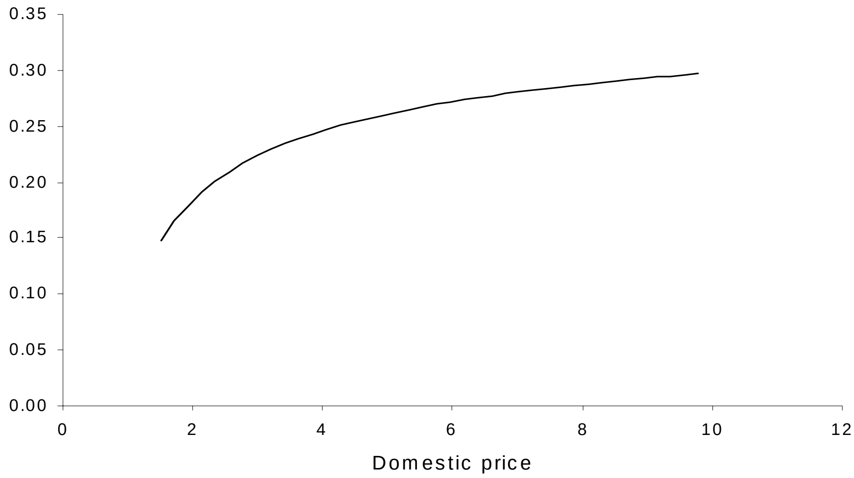

Figure 14 Aggregate Supply

6.2 The demand side of the economy

The demand-side is more complicated than the supply-side. There is two reasons for this. First, as typically in neo-classical models (with no non-produced good) the demand-side can only be analyzed *given* the supply-side. Observe, that the entire analysis in the last section could be done without any mention of the demand-side. For given domestic price, the supply-side comes recursively before the demand-side. Secondly, the behavior of the households, which is the most complicated part of the model, is to be found at the demand-side.

While the aggregate supply curve could be derived by solving a limited number of the models stationary state equations, the aggregate demand curve is found by solving the entire stationary state model (except the equilibrium constraint on the domestic good marked) for given domestic price. The entire model is therefore "behind" the aggregate demand curve.

The overall plan is as follows: first an aggregate consumption function is derived using the Keynes-Ramsey rule. This yields a linear relationship between aggregate consumption net of disutility of work and total wealth (non-human wealth plus human capital). Secondly, it is demonstrated that in stationary state human capital is a linear function of various sorts of labor income, inheritance and lump-sum transfers from the government. Finally the wealth accumulation equations are aggregated, and it is shown that in stationary state aggregate consumption net of disutility of work has to be equal to aggregate after-tax interest income plus non-interest income net of inheritance, such that there are no net savings.

For given domestic price the supply-side of the economy can be solved as shown above. Taking the supply determined variables as given and adding the tax equations, the 3 new aggregated equations can be used to calculate aggregate consumption, wealth and human capital for given domestic price such that aggregated demand for the domestic good can be calculated. In this way we have established a functional relationship between the domestic price and the aggregated demand for the domestic good. This is what we call the aggregated demand curve.

6.2.1 The aggregate consumption function

In this section we demonstrate the key result in this chapter that all though consumption can not be aggregated outside stationary state, this is in fact the case in stationary state. In stationary state consumption net of disutility of work can be aggregated to a standard linear consumption function a la Blanchard. The aggregation procedure involves 3 steps. First it is proven using the Keynes-Ramsey rule that in stationary state the consumption net of disutility of work for each generation is proportional with aggregate consumption. This property allows us to split aggregate consumption into the consumptions of the generations. Secondly, we use that for each generation there is a simple linear stationary state-relationship between consumption and wealth. We therefore has a linear relationship between aggregate consumption and the wealth of each generation. Finally we aggregate to get the linear relationship between aggregate consumption and aggregate wealth.

To deduce the aggregate behavior of the households we start from the individuals optimal choices. Using the Keynes-Ramsey rule, (4.37) p. 70, we can prove that in stationary state

$$Q_b = \alpha_b Q \tag{6.27}$$

where

$$\alpha_{18} \equiv \left(N_{18}^{EF} + \sum_{b=19}^{77} N_b^{EF} \left(\frac{1 + r\,(1 - t^r)}{1 + \theta} \right)^{S(b-18)} \right)^{-1} \tag{6.28}$$

and

$$\alpha_b \equiv \begin{cases} \alpha_{18}\left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S(b-18)} & \text{for } 18 < b < 78 \\ \xi^S \alpha_{18}\left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S\cdot 60} & \text{for } b = 78 \end{cases} \tag{6.29}$$

The proof is given in appendix C. (6.27) gives a full expression of household consumption in terms of aggregate consumption net of disutility of work. Note that for given stationary state-population, after-tax interest rate and rate of discount, the distribution of consumption net of disutility of work among households is given and independent of the rest of the parameters and variables of the model.

$$\eta_{b-1} \equiv \left[\sum_{i=b}^{78}\left(v_{b-1,i}\cdot N_i^{EF}\right)^S \left(P\tilde{R}_{b-1,i}\right)^{1-S}\right]^{\frac{1}{1-S}} \tag{6.30}$$

From (4.35) and (6.30) evaluated in the stationary state we have that for $18 \leq b \leq 78$

$$\begin{aligned} Q_b &= \xi_b^S \left(\frac{1+r_t\left(1-t_t^r\right)}{1+\theta}\frac{\eta_{b-1}}{P}N_{b-1}^{EF}\right)^S \frac{a_{b-1}+H_{b-1}}{\eta_{b-1}} \\ &= \xi_b^S \left(\frac{1+r\left(1-t^r\right)}{1+\theta}N_{b-1}^{EF}\right)^S \left(\Delta_{b-1}\right)^{S-1}\frac{a_{b-1}+H_{b-1}}{P} \end{aligned} \tag{6.31}$$

where

$$\eta_{b-1} \equiv P\left[\sum_{i=b}^{78}\left(v_{b-1,i}\cdot N_i^{EF}\right)^S \left(\tilde{R}_{b-1,i}\right)^{1-S}\right]^{\frac{1}{1-S}} \equiv P\Delta_{b-1} \tag{6.32}$$

Combining this with (6.27) leads to

$$\xi_b^S \left(\frac{1+r\left(1-t^r\right)}{1+\theta}N_{b-1}^{EF}\right)^S \left(\Delta_{b-1}\right)^{S-1}\frac{a_{b-1}+H_{b-1}}{P} = \alpha_b Q \tag{6.33}$$

such that

$$a_b + H_b = \xi_{b+1}^{-S}\left(\frac{1+\theta}{\left[1+r\left(1-t^r\right)\right]N_b^{EF}}\right)^S \left(\Delta_b\right)^{1-S}\alpha_{b+1}PQ \tag{6.34}$$

This implies that aggregate non-human wealth now can be written in terms of Q and aggregate human capital, H

$$A + H \equiv \sum_{b=18}^{77} N_b^{EF} a_b + \sum_{b=18}^{77} N_b^{EF} H_b = \beta^{-1}PQ \tag{6.35}$$

where

$$\beta \equiv \left[\sum_{b=18}^{77} N_b^{EF}\xi_{b+1}^{-S}\left(\frac{1+\theta}{\left[1+r\left(1-t^r\right)\right]N_b^{EF}}\right)^S \left(\Delta_b\right)^{1-S}\alpha_{b+1}\right]^{-1} \tag{6.36}$$

β is exogenous because it consists only of exogenous variables. By manipulating the expression of A in (6.35) we get a macro consumption function net of disutility from work

$$Q = \beta \frac{A + H}{P} \tag{6.37}$$

This is a standard consumption function a la Blanchard. The structure of the model implies that the constant, β, is a quite complicated function of the discount rate, the interest rate and the structure of population.

To calculate the "pure" macro consumption, C

$$C \equiv \sum_{b=18}^{77} N_b^{EF} C_b \tag{6.38}$$

we use equation (4.10) p. 63, the optimal labor supply given by (4.25) p. 67 and the expressions for Z_b and Q - (4.11) p. 63 and (6.37) respectively. This yields

$$\begin{aligned} C &= \sum_{b=18}^{77} N_b^{EF} Q_b + \sum_{b=18}^{77} N_b^{EF} Z_b = Q + \sum_{b=18}^{77} N_b^{WF} f(\ell) \\ &= \beta \frac{A+H}{P} + N^W f(\ell) \end{aligned} \tag{6.39}$$

where N_b^{WF} denotes the number of working individuals belonging to household generation b, and N^W is the total number of workers in the economy as defined below

$$N^W \equiv \sum_{b=18}^{77} N_b^{WF}, \quad \text{and} \quad N_b^{WF} \equiv N_b^{FW} + \sum_{a=18}^{60} N_a^M \cdot \omega_{a,b} \tag{6.40}$$

6.2.2 Aggregate human capital

We still need to determine the values of aggregate human capital, H. H is defined as

$$H = \sum_{b=18}^{77} N_b^{EF} H_b = \sum_{b=18}^{77} N_b^{EF} \sum_{i=b+1}^{77} \tilde{y}_i \tilde{R}_{b,i} \tag{6.41}$$

where $\tilde{y}_i$ is the income per adult in the household of age i net of disutility of work. In stationary state, the income per adult in the household of age b can from (4.3) p.

59 be written as (see Appendix C)

$$
\begin{aligned}
y_b \;=\;& \tfrac{N_b^{WF}}{N_b^{AF}}\left(\left[W-T^w\left(W\right)-b+T^b\left(b\right)\right]\ell+\left[b-T^b\left(b\right)\right]\bar{\ell}\right)\\
&+\tfrac{N_b^{AF}-N_b^{WF}}{N_b^{AF}}\left[f^P-T^P\left(f^P\right)\right] \qquad (6.42)\\
&+\tfrac{1}{N_b^{AF}}\left(N_b^{FW}b_b^{inF}+\sum_{a=29}^{60}N_a^M\cdot\omega_{a,b}\cdot b_a^{inM}\right)\alpha_{78}N_{77}^{EF}PQ\;+\bar{y}_b+\tau+\tau^W
\end{aligned}
$$

where $\bar{y}_b$ is the exogenously given generation specific transfers from the government

$$\bar{y}_b\equiv\frac{N_b^F}{N_b^{AF}}TR_b^F+\frac{1}{N_b^{AF}}\sum_{a=18}^{\tilde{A}}N_a^M\cdot\omega_{a,b}\cdot TR_a^M \qquad (6.43)$$

and

$$b_b^{inF}\equiv\frac{B_b^{inF}}{\alpha_{78}N_{77}^{EF}PQ}\;,\quad b_a^{inM}\equiv\frac{B_a^{inM}}{\alpha_{78}N_{77}^{EF}PQ} \qquad (6.44)$$

defines the shares of the total inheritance (which is $PN_{78}^{EF}Q_{78}=\alpha_{78}N_{77}^{EF}PQ$) that a female of age b and a man of age a receive. These shares can be considered constants for a stationary state population. The income adjusted for the disutility of work, $\tilde{y}_b$, given by (4.17) from page 65, can then be calculated given the income terms above

$$\tilde{y}_b=y_b\frac{N_b^{AF}}{N_b^{EF}}-PZ_b=y_b\frac{N_b^{AF}}{N_b^{EF}}-P\frac{N_b^{WF}}{N_b^{EF}}f\left(\ell\right) \qquad (6.45)$$

We are now able to deduce an expression for the aggregate human capital. Tedious calculations (see appendix) implies that

$$
\begin{aligned}
H \;=\;& f_1\left\{\left[W-T^w\left(W\right)-b+T^b\left(b\right)\right]\ell+\left[b-T^b\left(b\right)\right]\bar{\ell}-Pf\left(\ell\right)\right\}N^W \qquad (6.46)\\
&+f_2\left[f^P-T^P\left(f^P\right)\right]\left(N^A-N^W\right)+f_3\alpha_{78}N_{77}^{EF}PQ+\left(f_4\left(\tau+\tau^W\right)+\bar{h}\right)N^A
\end{aligned}
$$

where the help-variables f_1, f_2, f_3, f_4 and $\bar{h}$ are defined below

$$f_1\equiv\sum_{b=18}^{77}\frac{N_b^{EF}}{N^W}\sum_{i=b+1}^{77}\frac{N_i^{WF}}{N_i^{EF}}\tilde{R}_{b,i} \qquad (6.47)$$

$$f_2\equiv\sum_{b=18}^{77}\frac{N_b^{EF}}{N^A-N^W}\sum_{i=b+1}^{77}\frac{N_i^{AF}-N_i^{WF}}{N_i^{EF}}\tilde{R}_{b,i} \qquad (6.48)$$

$$f_3\equiv\sum_{b=18}^{77}N_b^{EF}\sum_{i=b+1}^{77}\left(\frac{N_i^{FW}}{N_i^{EF}}b_i^{inF}+\frac{1}{N_i^{EF}}\sum_{a=29}^{60}N_a^M\cdot\omega_{a,i}\cdot b_a^{inM}\right)\tilde{R}_{b,i} \qquad (6.49)$$

$$f_4\equiv\sum_{b=18}^{77}\frac{N_b^{EF}}{N^A}\sum_{i=b+1}^{77}\frac{N_i^{AF}}{N_i^{EF}}\tilde{R}_{b,i} \qquad (6.50)$$

$$\bar{h}\equiv\sum_{b=18}^{77}\frac{N_b^{EF}}{N^A}\sum_{i=b+1}^{77}\left(\frac{N_i^F}{N_i^{EF}}TR_i^F+\frac{1}{N_i^{EF}}\sum_{a=18}^{\tilde{A}}N_a^M\cdot\omega_{a,i}\cdot TR_a^M\right)\tilde{R}_{b,i} \qquad (6.51)$$

Equation (6.46) gives the stock variable, H (aggregate human capital), in terms of various flow variables. The translation from flow to stock is given by f_1, f_2, f_3, and $f_4$which are discounting factors dependent on the interest rate and the structure of population. H is given by the sum of 4 terms. The first term is proportional with labor income and unemployment benefits net of disutility of work. The second term is proportional with pensions. The third term is proportional with the value of total consumption net of disutility of work. This term measures the implication of inheritance. The fourth term is proportional with various sorts of transfers from the public sector

6.2.3 Aggregate savings

What remains to have a full characterization of the households aggregate behavior, is to aggregate equation (4.12) p. 65, which defines the accumulation of wealth. In stationary state

$$a_b = (1 + r(1 - t^r))\, a_{b-1}\frac{N_{b-1}^{EF}}{N_b^{EF}} + y_b \frac{N_b^{AF}}{N_b^{EF}} - PC_b$$

or

$$a_b = (1 + r(1 - t^r))\, a_{b-1}\frac{N_{b-1}^{EF}}{N_b^{EF}} + \tilde{y}_b - PQ_b \tag{6.52}$$

Therefore

$$\sum_{b=18}^{77} a_b N_b^{EF} = (1 + r(1 - t^r)) \sum_{b=18}^{77} a_{b-1} N_{b-1}^{EF} + \sum_{b=18}^{77} \tilde{y}_b N_b^{EF} - P \sum_{b=18}^{77} Q_b N_b^{EF}$$

or

$$A = (1 + r(1 - t^r)) \left(A - a_{77} N_{77}^{EF}\right) + \tilde{y} - PQ \tag{6.53}$$

where

$$\tilde{y} \equiv \sum_{b=18}^{77} \tilde{y}_b N_b^{EF} \tag{6.54}$$

From (6.52), and using the facts that $a_{78} = 0$, $N_{78}^{EF} = N_{77}^{EF}$ and $\tilde{y}_{78} = 0$

$$0 = (1 + r(1 - t^r))\, a_{77} - PQ_{78}$$

Substituting this into (6.53), and using that $Q_{78} = \alpha_{78} Q$ according to (6.27)

$$A = (1 + r(1 - t^r))\, A + \tilde{y} - P\left(Q + \alpha_{78} N_{77}^{EF} Q\right)$$

such that

$$r\left(1-t^{r}\right)A+\tilde{y}=\left(1+\alpha_{78}N_{77}^{EF}\right)PQ \tag{6.55}$$

Using (6.42), aggregate non-interest income net of disutility from work, $\tilde{y}$, can be calculated to

$$\begin{aligned}\tilde{y} &= \left(\left[W-T^{w}\left(W\right)-b+T^{b}\left(b\right)\right]\ell+\left[b-T^{b}\left(b\right)\right]\overline{\ell}-Pf\left(\ell\right)\right)N^{W}\\ &\quad+\left[f^{P}-T^{P}\left(f^{P}\right)\right]\left(N^{A}-N^{W}\right)+\ \alpha_{78}N_{77}^{EF}PQ\ +\bar{y}+\left(\tau+\tau^{W}\right)N^{A}\end{aligned} \tag{6.56}$$

where

$$\bar{y}=\sum_{b=18}^{77}\bar{y}_{b}N_{b}^{AF} \tag{6.57}$$

Substituting (6.56) into (6.55) yields

$$PQ=r\left(1-t^{r}\right)A+\tilde{y}^{\setminus inh} \tag{6.58}$$

where

$$\tilde{y}^{\setminus inh}\equiv\tilde{y}-\alpha_{78}N_{77}^{EF}PQ \tag{6.59}$$

According to (6.58) aggregate consumption in stationary state should be equal to interest income, $r\left(1-t^{r}\right)A$, plus non-interest income net of inheritance, $\tilde{y}^{\setminus inh}$, such that there are no net savings. Inheritance is not included as this is an inter-household transfer.

6.2.4 Aggregate demand

We can now use the 3 aggregated equations derived in the last 3 sections to calculate the aggregated demand. Adding the tax equations, and using (6.37), (6.46), (6.56) and (6.58) we have

$$\tau N^{A}=-r\bar{B}^{g}-\hat{g}\left(pY^{*}-M^{*}\right)-\bar{y}-b\left(\overline{\ell}-\ell^{*}\right)N^{W}-f^{P}\left(N^{A}-N^{W}\right)-FO+tax \tag{6.60}$$

$$\begin{aligned}tax &= T^{w}\left(W^{*}\right)\ell^{*}N^{W}\quad+\quad T^{b}\left(b\right)\left(\overline{\ell}-\ell^{*}\right)N^{W}\quad+\quad T^{P}\left(f^{P}\right)\left(N^{A}-N^{W}\right)\\ &\quad+t^{a}W^{*}\ell^{*}N^{W}\quad+\quad t^{VAT}\left(C^{f}+pC^{d}\right)\quad+\quad t^{r}r\left(A-V^{*}\right)\\ &\quad+t^{d}D^{*}\quad+\quad t^{c}\left(pY^{*}-\left(1+t^{a}\right)W^{*}\ell^{*}N^{W}-M^{*}-\hat{\delta}\hat{K}^{*}-rgpK^{*}\right)\end{aligned} \tag{6.61}$$

$$Q = \beta \frac{A+H}{P} \tag{6.62}$$

$$\begin{aligned} H &= f_1 \left\{ \left[W^* - T^w \left(W^* \right) - b + T^b \left(b \right) \right] \ell^* + \left[b - T^b \left(b \right) \right] \bar{\ell} - Pf \left(\ell^* \right) \right\} N^W \\ &+ f_2 \left[f^P - T^P \left(f^P \right) \right] \left(N^A - N^W \right) + f_3 \alpha_{78} N_{77}^{EF} PQ + \left(f_4 \left(\tau + \tau^W \right) + \bar{h} \right) N^A \end{aligned} \tag{6.63}$$

$$\begin{aligned} \tilde{y} &= \left(\left[W^* - T^w \left(W^* \right) - b + T^b \left(b \right) \right] \ell + \left[b - T^b \left(b \right) \right] \bar{\ell} - Pf \left(\ell^* \right) \right) N^W \\ &+ \left[f^P - T^P \left(f^P \right) \right] \left(N^A - N^W \right) + \ \alpha_{78} N_{77}^{EF} PQ \ + \bar{y} + \left(\tau + \tau^W \right) N^A \end{aligned} \tag{6.64}$$

$$PQ = r \left(1 - t^r \right) A + \tilde{y}^{\setminus inh} \tag{6.65}$$

$$\tilde{y}^{\setminus inh} \equiv \tilde{y} - \alpha_{78} N_{77}^{EF} PQ \tag{6.66}$$

$$f^p = \varphi^p W^* \left(1 - t^\ell \right) \tag{6.67}$$

$$b = \varphi^b W^* \left(1 - t^\ell \right) \tag{6.68}$$

$$C^d = \left(\frac{\left(1 + t^{VAT} \right) p}{A_c \mu_c P} \right)^{-E} \frac{C}{A_c} \tag{6.69}$$

$$C^f = \left(\frac{\left(1 + t^{VAT} \right)}{A_c \left(1 - \mu_c \right) P} \right)^{-E} \frac{C}{A_c} \tag{6.70}$$

$$P = \frac{1}{A_c} \left(1 + t^{VAT} \right) \left(\left(1 - \mu_c \right)^E + \left(\mu_c \right)^E p^{1-E} \right)^{\frac{1}{1-E}} \tag{6.71}$$

$$Q = C - N^W f \left(\ell^* \right) \tag{6.72}$$

where subscript * indicates that the variable is determined at the supply side of the economy. (6.60) and (6.61) defines the tax system. (6.62) is the aggregate consumption function. (6.63) defines aggregate human capital. (6.64)-(6.66) defines aggregate savings. (6.67) and (6.68) defines indexation of pensions and unemployment benefits. (6.69) and (6.70) translates aggregate consumption into consumption of domestic and foreign goods. (6.71) defines the price index, P, and (6.72) defines consumption net of disutility of work.

For given domestic price we have 13 variables (Q, A, H, P, $\tilde{y}$, $\tilde{y}^{\backslash inh}$, τ, tax, f^P, b, C^d, C^f, C) and 13 equations. For the current calibration one can verify numerically that the system has a unique solution for any positive value of the domestic price p. Let $C^d(p)$ denote the solution value of private consumption of the domestic good C^d for given domestic price. We can now define the aggregate demand

$$Y^D(p) \equiv C^d(p) + \delta K(p) + \hat{g}\frac{pY^S(p) - M(p)}{p} + X(p) + \frac{FO}{p}, \tag{6.73}$$

where $K(p)$, $Y^S(p)$ and $M(p)$ are given from the supply-side ((6.25), (6.24) and (6.26)), FO is exogenous and exports $X(p)$ is given by

$$X(p) = \chi p^{-\epsilon} \tag{6.74}$$

Solving the demand system for various values of p, and given the supply-side of the economy, results in figure 15

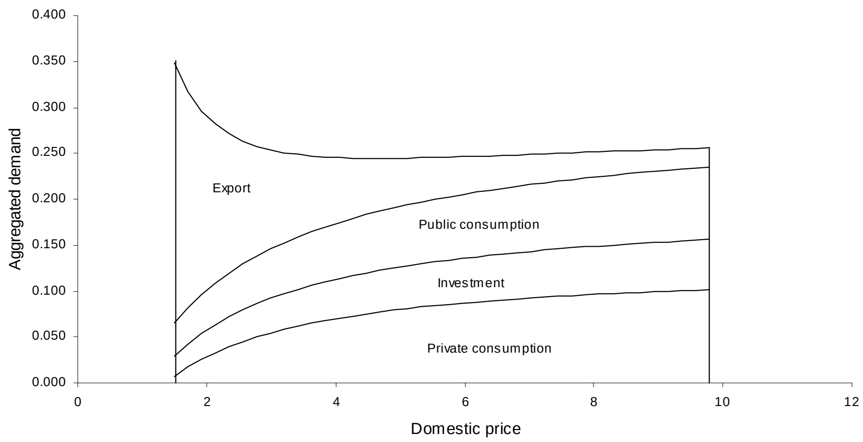

Figure 15 Agg. demand components

For the current calibration, the aggregate demand curve is first decreasing and then increasing. The reason for this is the interaction between the components of aggregate supply. For low domestic price the export is relatively high while the rest of the components is relatively low. For high domestic price the situation is the opposite: the export is low and the rest of the components are relatively high. The fact that public consumption and investment is increasing in the domestic price is explained from the supply-side. A high price implies a high level of activity and therefore high investments. The same explanation can be given concerning public consumption, as this is defined by $\hat{g}\left(Y^S - M/p\right) + FO/p$. Aggregate private consumption is increasing in p because both real wealth A/P and real aggregate human capital H/P is increasing in p.

6.2.5 Stationary state equilibrium

We can now demonstrate the uniqueness of the stationary state for the current calibration. The equilibrium is shown in figure 16

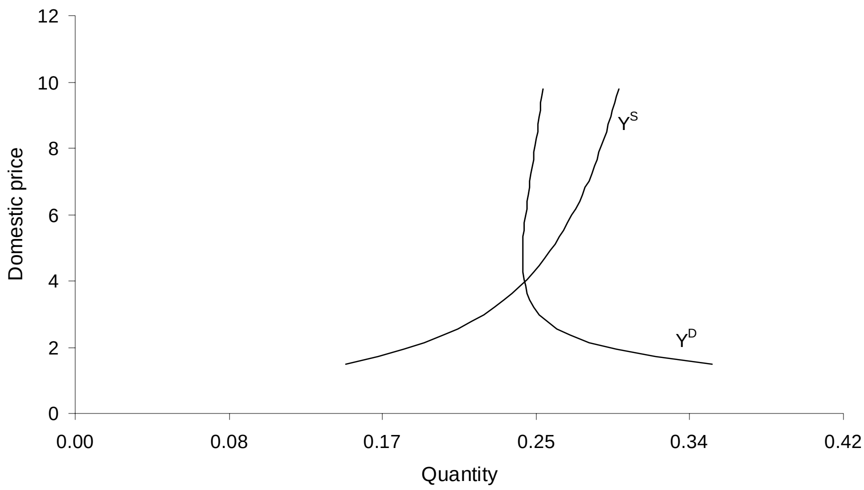

Figure 16 Aggregate supply and demand

While the supply curve looks very familiar, the demand curve is more interesting with its "forward-bending" character. As explained in the last section, the form is due to the interaction of exports and the other components of final demand. Observe that the demand curve is almost horizontal close to equilibrium. The equilibrium price is very close to the price that minimize demand.

6.3 Life cycle behavior in stationary state

In this section we describe the consumption and saving behavior of a household over its entire lifetime in a stationary state. The analysis is based on the numeric simulations.

6.3.1 Household size

Before we turn to the economic behavior of the households it is helpful to bear in mind the pattern of the household size in the planning period (as presented in chapter 3).

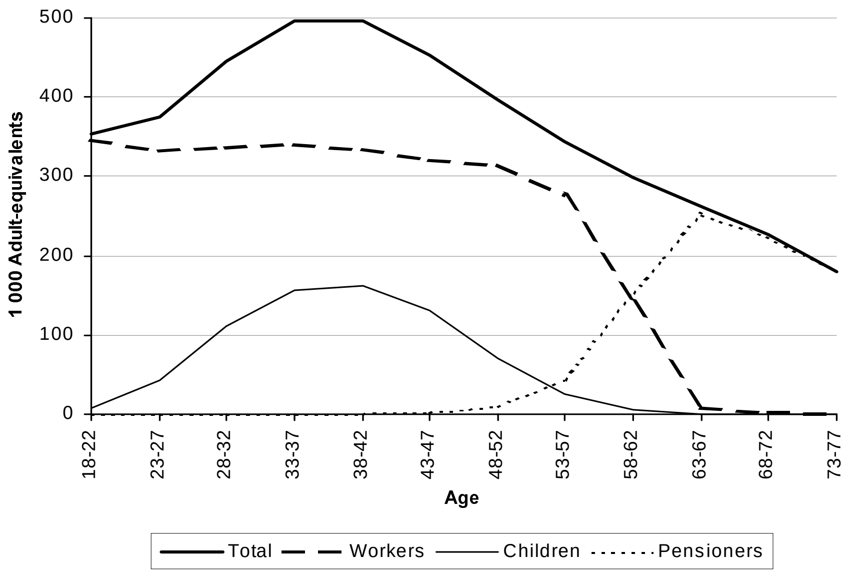

Figure 17 Household size

In figure 17 the total number of adult-equivalents in the household is shown. Furthermore it is split into its three components - namely: children, workers and pensioners.

As described earlier the number of mothers giving birth is maximized at the age of 28-29 and the distribution is almost symmetric around this point (cf. figure 5 at page 46). Nevertheless we observe that the number of children in the household first reaches its top level when the mothers are in their late-thirties. This is due to the fact that many women have more than one child. Later on as the children reach the age of 18 they leave home and the number of children start to decline as we see for the age group of 43-47 years and further on. At the age of 58-62 years practically all children have left the household since only a very small part of the women is above 40 years or above when giving birth.

The number of workers on the other hand is fairly constant for the first 35 years, then starts to decrease, and at the age of 63-67 is very close to zero. At this time all the women in the household have retired from the labor market, but since they are married to a distribution of men - who on average are three years older though - a small part of these men will be below the pension age of 61 years (and thereby still working). For the following 5-year groups it becomes more and more unlikely for a woman to be married to a working man and the number of workers in the household then approximates zero.

The pattern for the pensioners is just the opposite one. However we observe that pensioners begin to be present in the household already at the age of 48-52, and then the number rises as the workers start to retire. This goes on up to the age of 63-67 where the intake from this source ceases. Then a gradual decline takes over - simply as a result of bereavement.

The characteristics presented above are very important in the understanding of the household behavior, and will be referred to from time to time in the following.

6.3.2 Total household income and its composition

Also a description of the income and its composition, which is closely related to the demographics stated above, is needed before turning to describe the life cycle behavior. In figure 18 total disposable income of a household is split into capital (interest) income and non-interest income

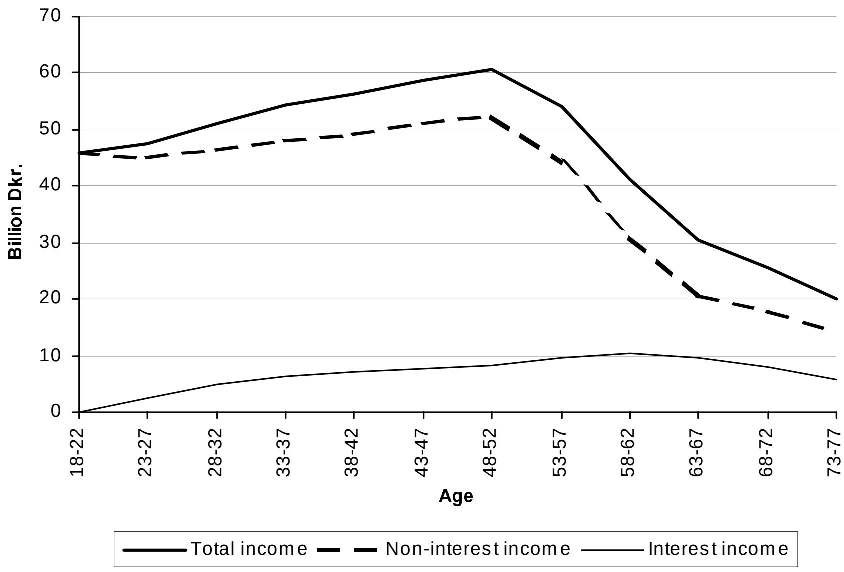

Figure 18 Total household income

Roughly speaking the total disposable income has increased linearly for the first 7 periods, i.e. from the age of 18 to the age of 52, and is thus altogether risen more than 30 percent. The main contributor to this pattern arises from an increase in the interest income as savings are being accumulated (this picture is confirmed later on). Also the non-interest income as a whole is growing during the first periods though. This is due to a rise in the amount of inheritance, as will be described more thoroughly later.

The total household income net of taxes is at its maximum at the age of 48-52. In the next 25 years the income falls dramatically to one third of the top level. This is caused by an almost identical reduction in the non-interest income which accounts for most of the income. Further this reduction is brought about from mainly two sources: first a gradual shift from a high share of labor participators to a high share of pensioners (who earns less) in the household and secondly from a reduction in the number of adults, caused by death.

The relative composition of the total income is shown in figure 19

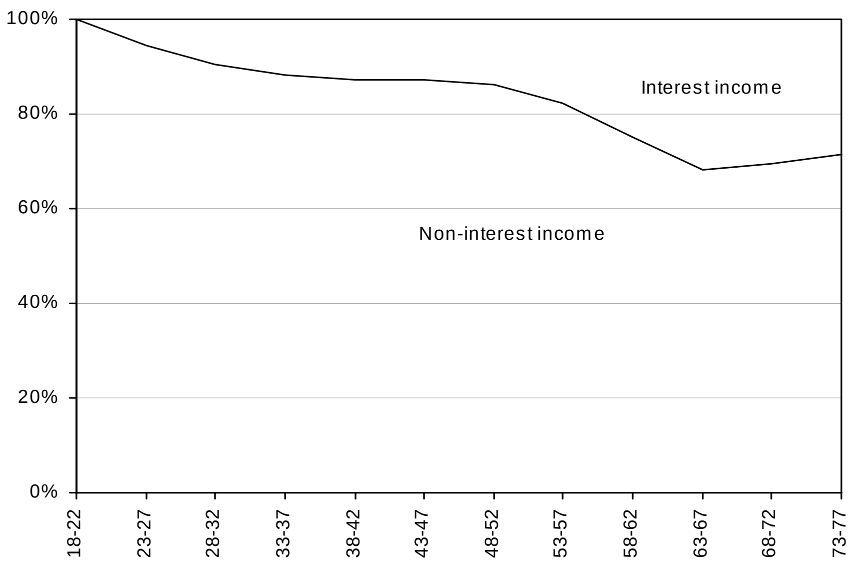

Figure 19 Relative composition of total household income

During the first 15 years of the household's existence more than 90 percent of the total income is non-interest income. As the stock of wealth gradually rise (due to persistent net savings) and thereby also the interest yields grow, the share of income that stems from interests becomes larger and larger. After 35 years the share is close to 20 percent and then as non-interest income subsequently commences the steep fall, the interest's share of income increases rapidly until a level of 32 percent is reached at the age of 63-67.

During the last 10 years the interest share out of total income is reduced slightly to 29 percent in spite of the continued drop in non-interest income. The reason is that interest income falls at an even greater pace because of the phasing out of the stock of wealth during the old age.

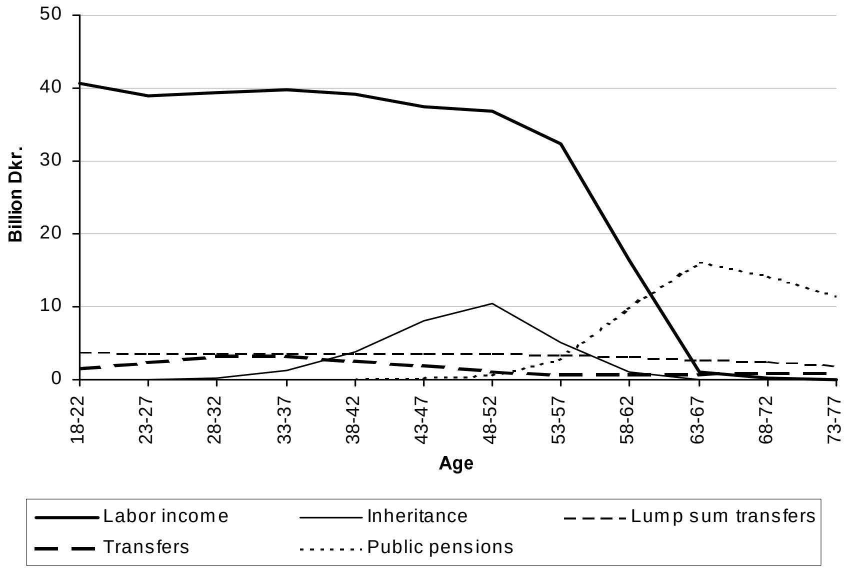

Figure 20 Non-interest household income

Figure 20 presents a closer look at the non-interest income, which is the sum of labor income (including unemployment benefit), inheritance, lump-sum transfers, exogenous age specific transfers and public pensions[1]. The division into these five categories is what we see in the figure. Note that all the taxable terms is net of taxes.

First of all we notice that labor income is sovereignly the biggest contributor to non-interest income until the age of 63-67, where the number of pensioners begins to dominate and the income arising from pensions becomes the single most important of the non-interest income categories.

We also note that the household is already receiving public pension at the age of 50, which of course is due to the fact that women are married to a distribution of men,

[1]Since the labor supply of a worker is constant during the working period - see (6.21) at page 94 - and thereby also the degree of unemployment, the fluctuations in the unemployment benefit will exactly match the ones of the labor income. Therefore these two income terms are pooled in the following analysis.

and at this stage the difference in age is "small" enough for some of the women to be married to men retired from the labor market. On the other hand the amount of labor income at the age of 63-67 is close to zero even though the average age is only 4 years above the pension age. At this level of course all the labor income has to come from younger men since the women are all retired, but because of the pattern for women to marry men, who on average are 3 years older, only a very small part of the women in the respective age group is matched with working men. This corresponds exactly to what we observed from figure 17.

The small fluctuations in the labor income during the first 20-30 years are (according to figure 10 at page 52) caused by the difference in the number of men per woman assigned to the household at different age levels. For instance the tiny rise in labor income at the ages of 28-37 is due to the fact that the household actually consists of more (working) people (men) as we saw in figure 17. The fall in the public pensions along the last two 5-year periods is solely a result of death among the members of the household.

Another point of the figure is the bell-shape of the inheritance that reaches its maximum at the age of 48-52. This is very reasonable since the most frequent ages for a mother to give birth are 28 and 29 years (according to figure 5 at page 46). This again means that her children will be 49-50 years old when she becomes 78 years old and the bequest is being paid out. Furthermore the distribution of mothers' age when giving birth has a similar bell-shape, which obviously is what is reflected in the pattern of the inheritance. However note how the average difference in age of about 3 years for a couple shows as a little left skewness in the curve for inheritance. When the men reach the most frequent age of receiving inheritance, namely 49-50 years as explained above, they will on average be married to women at the age of 46-47 years and this gives rise to the moderately different shape of the curve.

The last two components of the non-interest income are transfers. First, look at the lump-sum transfers, which are slightly decreasing in absolute value. This is simply because of the reduction in the household size since the lump-sum transfer per adult is not age dependent and remains constant in stationary state. Therefore the reduction speeds up in the last periods, where death intensity increases. Second, consider age specific transfers. Contrary to labor income, lump sum transfers and public pensions, these develops positively in the first 2 periods and are even doubled during this time. The rise is of course related to further educations and expenses associated with newborns, such as the costs of maternity leave, day care etc. Subsequently the age specific transfers gradually fall to a very low level at the retirement age. Note that these transfers are exogenous and rely on the data from the joint work of the Danish Ministry of Finance and EPRU - see Jacobsen et al. (1997).

In the next graph the percentage that each of the non-interest income terms contributes to the total non-interest income is shown, so that the five terms sum up to 100 percent.

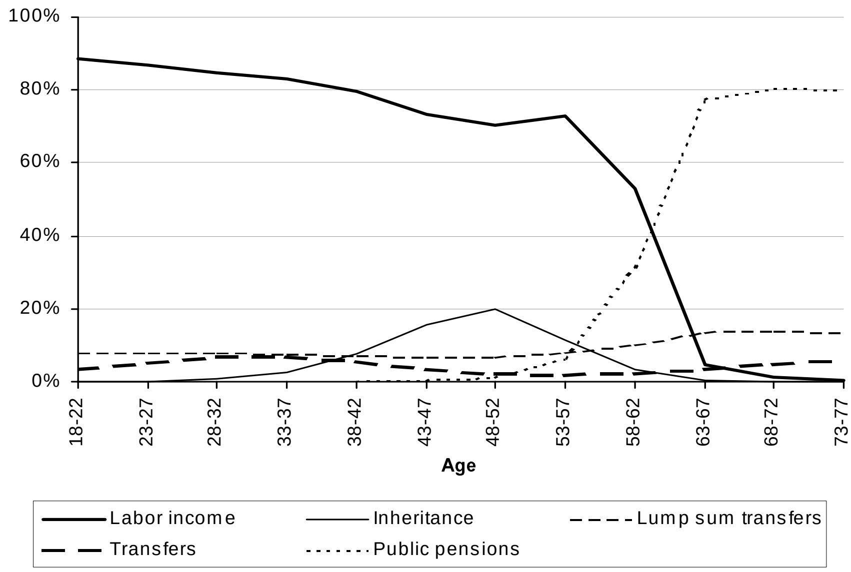

Figure 21 Relative composition of non-interest income

Again we see that labor income is the main income source during most of the household's time horizon and until the age of 53-57 accounts for between 70 and 89 percent of the non-interest income alone. Perhaps a bit surprising, the labor income still accounts for 53 percent at the age of 58-62 compared to a share of 31 percent arising from public pensions, in spite of the fact that the household at this stage to a larger extent consists of pensioners than of workers (remember that even though close to 60 percent of the women in this category will be workers, only about 35 percent of the men are working since they on average are 3 years older than their partner). The explanation is simply that the public pension net of taxes (per pensioner) is only 59 percent of the disposable wage and thereby also significantly lower than the unemployment benefit, which net of taxes corresponds to 77 percent of the disposable wage.

In period 9 and 10 (ages between 58 and 67) the labor income share falls drastically while the opposite picture holds for the public pensions. Thus, the last 15 years of the household's existence, pensions contribute by close to 80 percent of the non-interest income.

Now turning to lump sum transfers, we see that the share of non-interest income - arising from these - is constantly around 7-8 percent for the first 40 years. Then a strong increase sets in, and at the age of 63-67 lump sum transfers are doubled in relative size. This is not a token for a rise in the amount of lump-sum transfers (they are actually falling as mentioned earlier), but just a result of the huge drop in non-interest income in this age interval. The new level of 14 percent remains unchanged for the last three periods.

The age specific transfers are at their maximum level at ages between 23 and 42 and are in this period responsible for 5 to 7 percent of the non-interest income. Then it drops gradually to a level of only 2 percent when the household is in it's fifties. In the end there is a new increase however, and the transfers's share reach 6 percent in the last period - again the rise is due to the general fall in non-interest income.

Finally, when looking at the share of income that stems from inheritance we see a pattern similar to the analysis in absolute terms. The contribution of inheritance is first and foremost important at ages between 38 to 57 and reaches its top level of one fifth at the age 48-52. This is a quite considerable share and shows the importance of inheritance in the income formation.

This ends the description of the total household income and we now turn to measure the different income terms relative to the number of adults belonging to the household.

6.3.3 Income per adult

In this subsection we eliminate the effect on income owing to bereavement by looking directly on income per adult. This is pictured in figure 22

First of all we see a pattern very similar to the corresponding figure for the household as a whole (figure 18). The biggest difference is in the (relative) size of the fluctuations, which here are somewhat bigger in the rising phase till the age of 48-52, and much smaller henceforth, where the income is reduced. The fall in the total income is thus only 40 percent here, compared to the drop of two thirds, when the total household

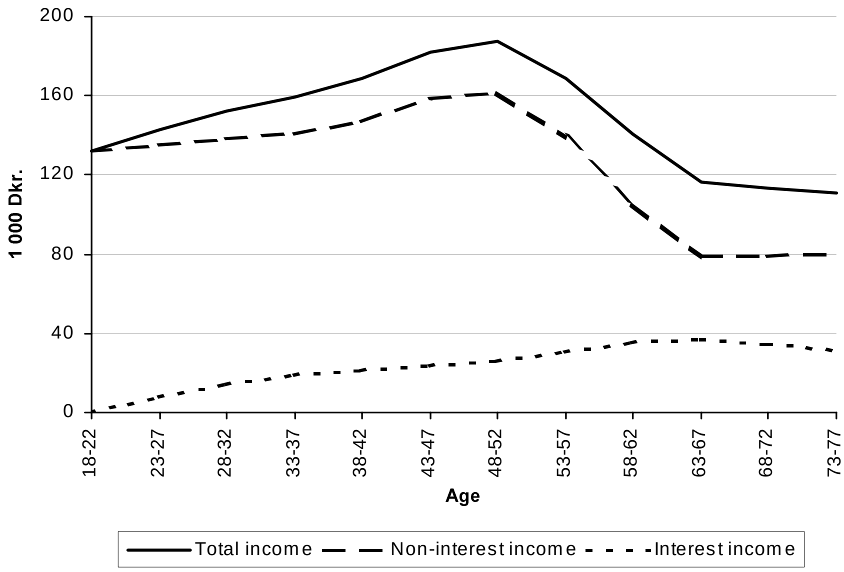

Figure 22 Total income per adult

income was considered.

We also note that the non-interest income is at the same level for the last three 5-year periods contrary to the continued decrease at the household level at this stage. Finally the interest income doesn't fall at the age of 63-67, even though savings are negative in the preceding 5-year period (as we shall see later on). This is off course because the smaller asset holdings are spread out on a lower number of adults such that assets per adult and thereby interests per adult are unchanged.

To get a more nuanced picture of the income formation, the non-interest income is split up into its main components as it was done earlier too

Figure 23 shows that labor income per adult is constant for the first 30 years. The reason is simply that all workers are paid the same - since there is no seniority fee and all supply the same amount of work - and the share of men who are pensioners up till this point is approximately zero (also remember that all the women are working). The following years labor income starts to decrease. Slowly at first and then very quickly as a larger and larger part of the adults becomes pensioners. This on the other hand gives rise to the observed increase in the average pension income per adult. Here we see a stabilization for the last 15 years, where the share of workers is the one to approximate zero. Also note, that the level for the pension is much lower than for the labor income as described earlier. Actually it constitutes only 55 percent of the

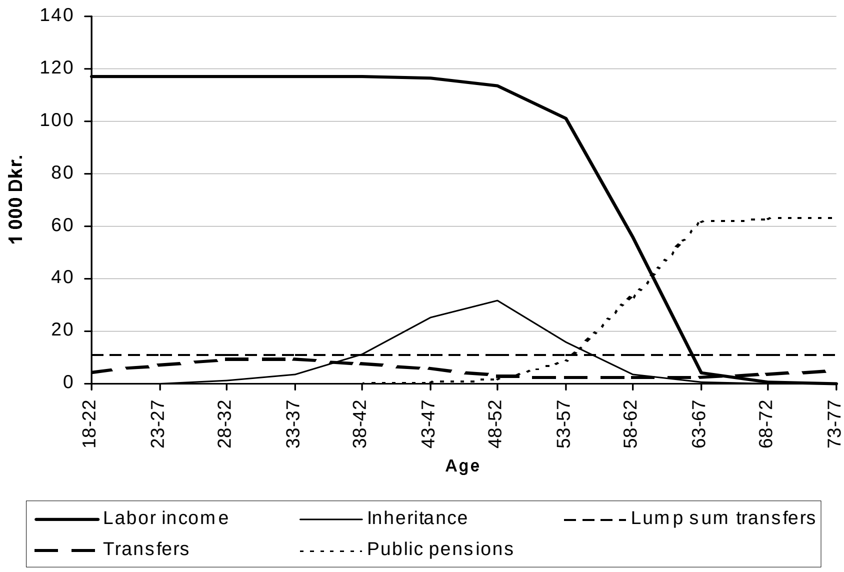

Figure 23 Non-interest income per adult

amount of labor income (including unemployment benefit).

The lump-sum transfers are constant for the whole life cycle since they are imposed directly on each adult and do not differ among individuals. Both the patterns for transfers and inheritance respectively follow the picture drawn earlier, and will not be stated again.

6.3.4 Household consumption and saving

We now look directly on the life cycle behavior for the household as illustrated by figure 24

The income curve is already explained and therefore we go on to describe the household consumption. For the first 20 years consumption increases steadily (along with the household size) and doubles in value during this period. In the next period, consumption manages to keep its level in spite of the considerable drop in the household size. The rest of the planning horizon is characterized by a moderate and almost linear reduction in consumption except for the last period , where the particularly big drop in household size takes place.

Note that savings, by definition, are determined as the difference between income and consumption. The savings are accumulated and are the cause for the constantly

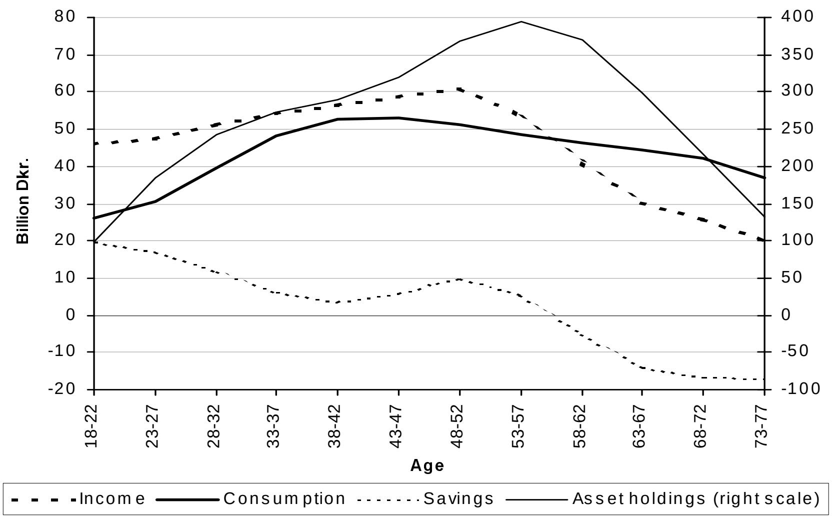

Figure 24 Life cycle household behavior (total)

growing amount of asset holdings up till the age of 53-57. At this point they have reached a level of 6-7 times the maximum yearly income level, but then a drastic fall follows.

The fall occurs because consumption starts to exceed the income and has to be financed by a cut in the asset holdings, which also appears in the figure as the negative savings in these periods. At the end of the last period the asset holdings are reduced by two thirds in comparison to its maximum level. The remaining assets at the age of 77 is the amount of bequest allocated to the heirs of the household.

When taking a closer look at the savings profile we observe that the household saves a lot during the first two periods (around 40 percent of their income), and then over the next 15 years savings drops to only 6 percent of the income (or 18 percent of the initial savings). This is of course associated with the costs of child rearing. At first the household is saving to meet the foreseen extra costs of having children and when the number of children later on are at its height savings are reduced, but even during this period, net savings are still positive. This however, is due to the big interest income stemming from the accummulated assets holdings, since the non-interest income alone at this point is not enough to finance the consumption. As the number of children begin to fall savings are increasing again for a while, and actually rise 10 percentage-points up till the age of 48-52.

6.3.5 Consumption and saving per adult-equivalent

Since the household seeks to maximize consumption per adult-equivalent it is natural to look at the life cycle behavior in terms of per adult-equivalent as illustrated in figure 25

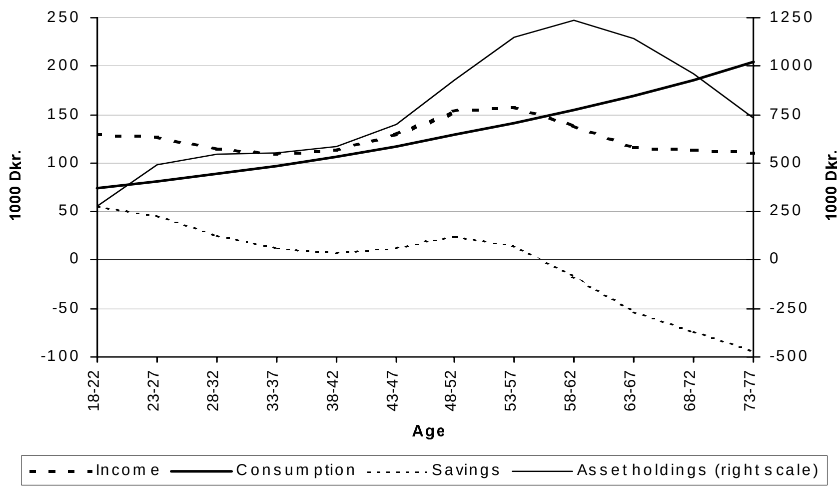

Figure 25 Life cycle behavior (per adult-equivalent)

The income curve has two tops now - in the first two periods and in period 7 and 8. Because only adults receive income the children only contributes to reduce the income per adult-equivalent. The effect from this is of course most evident when the household is in its thirties (where the number of children is highest). For the last 15-20 years the curve is practically identical to the one drawn per adult (see figure 22), since the number of children in this end of the scale is negligible.

The consumption pattern is also quite different from the pattern for the household as a whole. Consumption per adult-equivalent follows an approximately constant growth rate all the way through and at the end reaches a level 2.8 times the initial level (just above 200,000 Dkr.). This is the Keynes Ramsey rule in effect (cf. equation (4.37) at page 70 - in a stationary state version). It says in the stationary state that consumption corrected for disutility of work will grow at a constant rate provided that the interest rate after tax exceeds the pure rate of time preference and fall if the opposite is the case. In this calibration the former situation is the relevant one. Since the real value of the disutility from work is extremely low relative to that of consumption (due to a low elasticity of labor supply with respect to real reward, γ)[2], the growth rates of C and Q from generation to generation is almost identical.

With respect to savings the pattern with the two humps - as we saw in figure 24 in the last sub section - is still present although the relative fluctuation has increased. The reasoning of the saving decision is as before mentioned, that the household takes into account the number of children they are going to have, and save a lot during the first years to be able to cover the foreseen expenses. When the children eventually leave home, additional room for savings is made, and the savings increase until the income begins to drop dramatically as the household retires from the labor market. From age 58-62 and onwards savings per adult-equivalent have become negative.

When comparing the curve for the asset holdings with the corresponding one for the household as a whole we note some differences in the shape. First, the maximum level is dislocated to the following period (58-62) and the subsequent fall is not nearly as powerful as the first case indicates. The dislocation occurs since the fall in the total savings is surpassed by the reduction in the number of adult-equivalents. This makes it possible for the remaining individuals to have negative net savings per individual and still end up with bigger asset holdings (because of the take over from the people leaving the household). The less steep fall in the end is very much on the same grounds. Because of the reduction in number of adult-equivalents the total household asset holdings are spread out over fewer individuals and asset holdings measured per adult-equivalent wont have to decline as much. Second, the curve is much flatter when the household is in its thirties. This is due to the rise in the number of children.

This conclues the description of life cycle behavior in the stationary state.

[2]The disutility of work is equal for all workers in the stationary state (since they all supply the same amount of labor) and is - from (4.22) and (4.26) at page 67 - given by $f(\ell) = \frac{\gamma}{\gamma+1}\gamma_1\left(\frac{[W-T^w(W)]-[b-T^b(b)]}{\gamma_1 P}\right)^{\gamma+1}$. Now letting γ go to zero, the limit of $f(\ell)$ is seen to be zero and the disutility will vanish from the expression. But even for small values of γ, $\gamma+1$ can be approximated by 1 and $f(\ell)$ reduces to $\gamma\frac{[W-T^w(W)]-[b-T^b(b)]}{P}$, which will be quite small for a low value of γ (compared to the real value of consumption).

Chapter 7
CALIBRATION

Calibration is the process where numerical values are assigned to the parameters of the model.

Single country static CGE-models are calibrated to a data set for a specific country in a specific year. This is typically done by constructing a Social Accounting Matrix, which matches the markets and the budget constraints of the agents in the theoretical model. The well-known problem that data for investments, savings, and interest payments have to be fitted into a theoretical model, where such variables are not defined, has to be dealt with by manipulating either the data set or the behavioral relations of the model. One way of dealing with the problem, is to assume that the static equilibrium is in fact a stationary (or a steady) state of an underlying dynamic model.[1]

The major difference between dynamic models such as the present and static models is that in the dynamic model it is not possible to abstract from the accumulation process of physical capital and the intertemporal budget constraints of households. The analogy that comes closest to calibration of a static model is calibration of the stationary state of the dynamic model. However, the demands from the increased consistency requirements of the stationary state of a dynamic model lead to the problem that no base year satisfies all (if any) of the formal conditions for being a stationary state.

The solution applied to this problem is simple: Manipulate the actual data for the base year (the base year data set) so that the result (which we name the benchmark data set) satisfies the formal conditions for a stationary state and then calibrate the parameters on this benchmark data set by the usual procedure of inverting the relevant model equations. The benchmark data set might heroically be interpreted as

[1] Petersen (1997) gives an introduction to the calibration of static CGE models.

depicting the stationary state which would have prevailed in the base year, "if the base year were a steady state".

For the present dynamic theoretical model a unique non-trivial stationary state exists, if the population is stationary, as shown in chapter 6. Accordingly, one of the necessary manipulations consists of assuming a stationary population. The remaining manipulations are about resetting a flow variable (e.g. net investments) such that the corresponding stock variable (e.g. the capital stock) does not change. Finally, figures have to be imputed for some variables for which actual data do not exist or deviate too much from the theoretical concept.

Section 7.1 gives a short overview of all the steps involved in the entire calibration procedure. Section 7.2 briefly describes the base year data set utilized. The quantitative outcome of the calibration is listed in section 7.3. The remaining sections explain how this outcome is achieved.

As stated in section 7.1, there are several steps involved in the entire calibration procedure. To clarify the methodology, section 7.4 first goes through the details of the simplest part of the calibration procedure, namely the calibration of the parameters of the simple static equations relying on the base year data, by examining an illustrative example: The calibration of the export demand functions. The methodology is analogous to the calibration of static models.

Section 7.5 documents how (some of) the data are manipulated such that the resulting data set - which is named the benchmark data set - formally satisfies the requirement of a data set depicting a stationary state economy. Section 7.6 deals with the calibration of the remaining parameters on the benchmark data set. Finally, section 7.8 briefly outlines how the computer version of the model is solved.

7.1 A brief overview of the entire calibration procedure

The calibration procedure involves the following 4 steps:

1. Choosing a base year and collecting the base data set.

1995 is chosen as a base year. The core of the base data set is an input output table for the base year, but other statistics are utilized as well.

3. Constructing the benchmark data set from the base data set.

Inspecting the base year data reveals that there are several formal obstacles to regarding the base data set as generated by a stationary state economy. The base year data set is therefore manipulated so that the resulting data set - which is called the benchmark data set - satisfy the formal conditions for representing a stationary state solution to the model.[2] Most of these data manipulations are about altering the size of a flow variable (e.g. net investments by manipulating the physical depreciation rate) such that the corresponding stock variable (e.g. the capital stock) does not change during the benchmark year as required in stationary state, but also a synthetic stationary state population is compiled. Finally, data for some model variables - which either can not be observed or only can be measured with too little reliability - are imputed on model consistent assumptions.

3. *Calculate the models parameters so that model's stationary state solution reproduces the benchmark data set exactly.*

The parameters are calculated by inverting the models equations, i.e. by fixing the variables at their benchmark values and then solve the equations for the unknown parameters. With some exceptions (the intertemporal parameters in the utility function of households) this is done for one equation at the time. The procedure leaves some degrees of freedom as the model contains more parameters than non-identity equations. This leaves room for fixing in advance parameters on which there exists good prior information as for example behavioral/technical parameters such as elasticities of substitution, while more trivial parameters such as CES distribution parameters are calculated by solving the inverted equations. However the destinction between these two categories of parameters is not perfectly clear cut.

4. *If desirable: Constructing the benchmark data set II from the benchmark data set.*

This is done by reversing some of the former data manipulations if possible and desirable, using the model to calculate the new data set. For example the physical rate of depreciation in the model can be shocked back to its unmanipulated level and the resulting model solution can be regarded as the final benchmark data set II.

The calibrated model should be thought of as a parametrized theoretical model and the simulations as a way of exploring the model properties in a situation where the

[2] As point 4 below reveals, this benchmark data set should properly be named the benchmark data set I. For convenience it is however just referred to as the benchmark data set.

proportions of the model resembles those of the Danish economy. However, the present calibrated version should not be regarded as an empirical model of the actual Danish economy. For example, the population is assumed to be stationary. Comparing the figures **??** and **??** (page **??**) reveals that this is "rather far" from being the case.

7.2 The base year data set

In this version, 1995 is chosen as the base year. The actual base year data set is documented in details in appendix E. In this section a brief overview suffices. The actual data utilized are mainly national accounts data:[3]

1. Input output data.

A standard input output table in current prices is compiled for the base year.

2. Data for the capital stock.

The capital stock is defined as the so-called net capital stock of the national accounts. The net capital stock is the physical capital stock valued in depreciated replacement prices which take account of the remaining lifetime of each capital good. The private sector capital stock includes the capital stock of the whole private sector inclusive dwellings.

3. *Data for the public sector budget and net debt.*

The public sector is defined as the state (including the Social Pension Fund and the Central Bank) and local governments i.e. excluding the public funds (consisting mainly of ATP, AUD, LD and LG) which primarily administrate forced private pension schemes. These funds are classified as part of the public sector in the national accounts because their revenue is generated from tax-like arrangements. In the model, they are categorized as part of the private sector because the assumptions of perfect foresight and perfect capital markets make the forced private pension-savings of these funds easy to compensate for by voluntary savings/loans of the private agents.

4. *Data for the current account and the net foreign assets.*

[3] They are based on the old version of the national account. In October 1997, Statistics Denmark published a new national account based on the ENS95 standard. This will of course be utilized in the future calibration of DREAM. At present the new version of the national account is not yet complete and therefore the old version is utilized instead.

The current account and the net foreign assets are defined for the whole Kingdom of Denmark including the Faroe Islands and Greenland as opposed to the national account concepts.

All amounts are measured in billions of Dkr. The population is measured in thousands, i.e. per capita figures are measured in millions of Dkr. We apply the Harberger convention of measuring quantities such that the net prices of all goods are equal to one in the base year data set.[4] The import price including custom and import duties measured in Dkr., which is the standard numeraire of the model, is accordingly put equal to one. Also the wage costs inclusive the payroll tax, $W_{t_0}(1+t^a_{t_0})$, is put equal to one.

7.3 Results

The model is solved for 5 year periods to reduce the computational burden. This implies that flows are 5 time larger than for an annual frequency, while stocks are unchanged. Moving from an annual to a 5 year frequency, we however adopt the convention of keeping flows unchanged, and correspondingly dividing stocks by 5 to keep the relation between stocks and flows correct. Also, interest rates, discount rates etc. should be modified when moving from an annual to a 5 year frequency. This is done by multiplying by 5. This practice implicitly assumes that for example interests are only calculated once during the period, namely ultimo the period.

As it will be demonstrated in the next sections, the model contains enough parameters to leave room for fixing some of them freely in advance. They are displayed in table 7.1.[5] It is attempted to fix those parameters *a priori* which are especially important for the marginal properties of the model and for which some empirical knowledge exists. The table indicates whether and how a parameter should be modified when the frequency changes.

[4] The net price of a good is defined as the market price minus excise taxes and plus subsidies.

[5] The table excludes data for the population documented in details in chapter 3 and data for age dependent transfers distributed at age and gender documented in appendix E.

Parameter	Sector	Description	Value
PHIB		Unemployment benefits indexation	0.680
TWS		Average wage tax rate paid by the households	0.427
TLS		Labor market contribution tax rate	0.060
TUBS		Average tax rate on unemployment benefits	0.306
TTRANSPS		Tax rate on pensions	0.231
TDS		Tax rate on dividend income	0.349
TRS		Tax rate on interest income	0.477
TGS		Tax rate on capital gains	0.241
TCS		Corporate tax rate	0.340
TWAS		Wage tax rate paid by the employer	0.000
DTS		Tax rate on dividend income	0.719

2. Behavioral and technical parameters

Parameter	Sector	Description	Value
N		Rate of growth	0.000
IRS		Interest rate	0.250
THETA		Time preference	0.013
ALASTSHARE		Last period assets (bequest) as a share of all assets	0.040
GAMMA		Elasticity of labour supply	0.100
SIGMA_C		Elasticity of substitution (P-G) in consumer CES allocation	1.100
SIGMA_CP		Elasticity of substitution (PD-PF)in consumer CES allocation	1.500
SIGMA_Y	PRI-P	Elasticity of substitution (M(K-L))	0.250
SIGMA_Y	GOV-P	Elasticity of substitution (M(K-L))	0.250
SIGMA_YH	PRI-P	Elasticity of substitution (K-L)	0.600
SIGMA_YH	GOV-P	Elasticity of substitution (K-L)	0.600
SIGMA_YM	PRI-P	Elasticity of substitution (MP-MG)	0.100
SIGMA_YM	GOV-P	Elasticity of substitution (MP-MG)	0.100
SIGMA_YMP	PRI-P	Elasticity of substitution (MPD-MPF)	1.200
SIGMA_YMP	GOV-P	Elasticity of substitution (MPD-MPF)	1.100
SIGMA_I	PRI-P	Elasticity of substitution (ID-IFO)	1.500
SIGMA_I	GOV-P	Elasticity of substitution (ID-IFO)	1.300
EPSILON	PRI-P	Export demand elasticity	-1.400
EPSILON	GOV-P	Export demand elasticity	-1.400
D	PRI-P	Physical rate of depreciation of K	0.290
D	GOV-P	Physical rate of depreciation of K	0.187
DSK	PRI-P	Corporate debt share of the repl.value of K	0.600
PSI		Convexity of adjustment costs in prod. func.	2.000

Table 7.1. Parameters calculated *a priori*

The elasticities of substitution in the production function are taken from the Danish static CGE-model GESMEC, where the choice is based on a survey of Danish and international empirical studies, cf. Frandsen, Hansen & Trier (1995) with the exception of the elasticity of substitution between privately and governmentally produced materials, on which there exists scarce evidence. It is assumed to take the very moderate value of 0.1. The foreign trade elasticities are taken from the empirical estimates of the Danish macroeconometric models ADAM, cf. Dam (1995), and SMEC, cf. Det økonomiske Råds Sekretariat (1994), converting the import price elasticities of SMEC to the relevant elasticities of substitution of DREAM (9.33), cf. appendix E. The implicit aggregate own price elasticity of imports is equal to -1.0.

The labor supply elasticity is taken from the estimate of Smith (1995). The installation costs parameter, ι, is set equal to one which is convenient, because then the derivative of the installations costs with respect to either investments or the capital stock then only depend on the investment-capital ratio, cf. (2.22) and (2.23). The rate of time preference is fixed such that the asset profile of generations roughly mirrors micro evidence.

The listed tax rates are not defined as macro tax rates (aggregate revenue divided by aggregate base). Instead we aim at measuring their incentive impacts relevant in the optimization problems of the households and firms. Applying these parameter values may therefore deliver tax revenues deviating from the base year revenues. This has implications for the lump sum transfers which are calculated residually to balance the public budget, cf. section 7.5.1. The calculation of the tax rates and the unemployment indexation parameter is documented in details in appendix E.

The remaining parameters are listed in table 7.2. They are calculated residually under the restriction that the relevant equations apply, i.e. that they reproduce the benchmark data set. The next sections describe the procedure.

1. Policy parameters

Parameter	Sector	Description	Value
IOTA1		Public consumption relative to nominal GDP at factor costs	0.294
KAPPA		Public investments relative to nominal GDP at factor costs	0.024
TRANSPS_0		Average pension for each pensioner inclusive very olds persons	0.089
TVATS		Value added tax rate	0.164
TCPS		Excise quantity tax rate on consumption of the private good	0.071
TCGS		Excise quantity tax rate on consumption of the governmental good	0.071
TIS	PRI-P	Tax rate on investment goods	0.071
TIS	GOV-P	Tax rate on investment goods	0.259
TMS	PRI-P	Tax rate on material goods	0.013
TMS	GOV-P	Tax rate on material goods	0.242
TXS	PRI-P	Tax (subsidy) rate on export goods	-0.014
TXS	GOV-P	Tax (subsidy) rate on export goods	-0.014
TARIFFS		Tariff rate on foreign goods	0.007

2. Behavioral and technical parameters

Parameter	Sector	Description		Value
NY		Intertemporal elasticity of substitution		0.942
XI		Preference for leaving bequest		0.600
GAMMA1		Level of disutility of labour supply		1935430
LBAR		Maximum labor supply from a generation		0.209
PHIP		Public pensions indexation		0.095
MY_CP		Distribution factor in CES-aggregate of C	(Private)	0.959
MY_CG		Distribution factor in CES-aggregate of C	(Governmental)	0.060
MY_CPD		Distribution factor in CES-aggregate of CP	(Domestic)	0.851
MY_CPF		Distribution factor in CES-aggregate of CP	(Foreign)	0.358
MY_YM	PRI-P	Distribution factor in (MH) CES prod.func.	(Materials)	0.045
MY_YM	GOV-P	Distribution factor in (MH) CES prod.func.	(Materials)	0.007
MY_YH	PRI-P	Distribution factor in (MH) CES prod.func.	(H-aggregate)	0.081
MY_YH	GOV-P	Distribution factor in (MH) CES prod.func.	(H-aggregate)	0.195
MY_YHK	PRI-P	Distribution factor in (KL) CES prod.func.	(Capital)	0.389
MY_YHK	GOV-P	Distribution factor in (KL) CES prod.func.	(Capital)	0.032
MY_YHL	PRI-P	Distribution factor in (KL) CES prod.func.	(Labor)	0.315
MY_YHL	GOV-P	Distribution factor in (KL) CES prod.func.	(Labor)	0.911
MY_YMP	PRI-P	Distribution factor in CES-aggregate of M	(Private)	0.877
MY_YMP	GOV-P	Distribution factor in CES-aggregate of M	(Private)	0.445
MY_YMG	PRI-P	Distribution factor in CES-aggregate of M	(Governmental)	0.000
MY_YMG	GOV-P	Distribution factor in CES-aggregate of M	(Governmental)	0.000
MY_YMPD	PRI-P	Distribution factor in CES-aggregate of MP	(Domestic)	0.790
MY_YMPD	GOV-P	Distribution factor in CES-aggregate of MP	(Domestic)	0.883
MY_YMPF	PRI-P	Distribution factor in CES-aggregate of MP	(Foreign)	0.311
MY_YMPF	GOV-P	Distribution factor in CES-aggregate of MP	(Foreign)	0.154
MY_ID	PRI-P	Distribution factor in CES-aggregate of I	(Domestic)	0.853
MY_ID	GOV-P	Distribution factor in CES-aggregate of I	(Domestic)	0.927
MY_IF	PRI-P	Distribution factor in CES-aggregate of I	(Foreign)	0.356
MY_IF	GOV-P	Distribution factor in CES-aggregate of I	(Foreign)	0.162
CHI	PRI-P	Export demand level		305.003
CHI	GOV-P	Export demand level		0.212
PHI	PRI-P	Controls size of adjustment costs in prod. func.		1.138
PHI	GOV-P	Controls size of adjustment costs in prod. func.		0.000

Table 7.2. Parameters calculated residually

7.4 An example of the calibration method: calibration of the export demand functions

In this section the method of calibration of the parameters on the base year data set is illuminated by a thorough discussion of an example: the calibration of the export demand functions. The calibration is performed under the requirement that the base year values of the endogenous variables of the model are exactly reproduced by the calibrated equations.

The export demand functions (??) are restated below as

$$X_{t_0}^i = \chi^i \left(\left(1 + t_{t_0}^{iX}\right) p_{t_0}^i\right)^{-\varepsilon^i} \qquad i = P, G \tag{7.1}$$

where subindex t_0 denotes the benchmark year ($t = t_0$).[6] The calibration of this function is about assigning numbers to the variables (export demands, $X_{t_0}^i$, and export prices, $p_{t_0}^i$) and the parameters (the indirect export tax rates, $t_{t_0}^{iX}$, the numerical price elasticities, ε^i, and the scale parameters, χ^i).

First, it should be noted that the indirect tax parameters also occurs in (5.35) which for $t = t_0$ says

$$XT_{t_0} = \sum_{i=P,G} t_{t_0}^{iX} p_{t_0}^i X_{t_0}^i \tag{7.2}$$

where XT_{t_0} is the benchmark revenue of export taxes net of subsidies. Accordingly, calibration of (7.1) presupposes that (7.2) has been calibrated beforehand.

Therefore the calibration of (7.2) will be discussed first. The values of the variables in question in the base year data set can be read directly from the base year input output table.[7] First, according to the Harberger convention the units of measurement are defined such that all net prices (market prices net of excise taxes and subsidies) are equal to one in the base year data set, i.e. $p_{t_0}^i = 1$, $i = P, G$. This is very convenient, because it implies that exports at net prices in value terms are equal to exports in quantity terms. Therefore the quantity of exports in the base year data set can be read directly from the input output table as the deliveries from domestic production to exports, i.e. $X_{t_0}^P = 311.187$ billion Dkr. and $X_{t_0}^G = 0.216$ billion Dkr. measured in base year net prices. The revenue of excise taxes minus subsidies concerning exports can also be read directly from the export column as the delivery from net commodity taxes to exports, i.e. $XT_{t_0} = -4.433$ billion Dkr. The negative sign shows that it is a subsidy in net terms. For simplicity it is assumed that the export tax rates are equal for both exports items, i.e. $t_{t_0}^{PX} = t_{t_0}^{GX}$.

The requirement that the calibrated version of (7.2) should exactly reproduce the base year export tax revenue imposes one restriction (namely the equation (7.2) itself) on

[6] In this section, the benchmark data coincides with the base year data.

[7] The standard input output table is displayed as table 1 of appendix E. An expanded version including installations costs - which do not affect the export column - is shown as table 7.5 below.

the calibrated parameters of the equation. Since there is virtually only one parameter, $t_{t_0}^{PX} = t_{t_0}^{GX}$, it is therefore determined by this restriction as

$$t_{t_0}^{PX} = t_{t_0}^{GX} = \frac{XT_{t_0}}{\sum_{i=P,G} p_{t_0}^i X_{t_0}^i} = \frac{XT_{t_0}}{\sum_{i=P,G} X_{t_0}^i} = \frac{-4.433}{311.187 + 0.216} = -0.014236$$

as can also be seen in table 7.2. In this instance, there are no degrees of freedom in the calibration of the parameter.

We now turn back to (7.1). Consider the equation for private sector exports ($i = P$). The requirement that the calibrated version of (7.1) for $i = P$ should exactly reproduce the level of private exports in the base year data set, imposes one restriction on the calibrated parameters (namely the equation (7.1) for $i = P$ itself). Two parameters, ε^P and χ^P, remains to be calibrated, but there is only one restriction (7.1) for $i = P$, i.e. there is one degree of freedom. In this instance it is obvious that the degree of freedom should be utilized to fix the numerical export price elasticity, ε^P, since it is a parameter of great importance to the marginal properties of the model and it is also empirically well investigated. In contrast, the scale parameter, χ^P, is a much more "trivial" parameter whose function is to make things fit together (i.e. hit the actual level of exports) and having smaller consequences for the marginal properties of the model. We choose to set $\varepsilon^P = 1.4$, cf. table 7.1, because this is the long run estimate for total exports in the two Danish macroeconometric models ADAM, cf. Dam (1995), and SMEC, cf. Det økonomiske Råds Sekretariat (1994), although it seems to be a rather small value for a small open economy.[8]

Given $\varepsilon^P = 1.4$, the scale parameter, χ^P, can be calculated residually by solving (7.1) for χ^P

$$\chi^P = X_{t_0}^P \left(\left(1 + t_{t_0}^{PX}\right) p_{t_0}^P \right)^{\varepsilon^P} = 311.187 \cdot ((1 - 0.014236) \cdot 1)^{1.4} = 305.003$$

cf. table 7.2. Inserting the calibrated variables and parameters at the right hand side of (7.1) assures that the equation exactly reproduces the calibrated value of the left hand side variable as required. Of course, the export demand function of the governmental output is calibrated similarly.

Comparing this calibration procedure with that of the traditions of macroeconometric models like ADAM and SMEC, two differences show up.

[8] To be exact, it is equal to -1.5 in SMEC.

First, the "free" parameter, ε^P, can be fixed with reference to all kinds of empirical studies, whereas the traditional approach of macroeconometric modellers is to estimate all the parameters of (7.1) on the historical time series of the variables $X_{t_0}^P$ and $p_{t_0}^P$. The value of the "free" parameter(s) can for example be chosen from studies using other data as for instance very disaggregated time series data, cross section data or panel data.

Second, assuming that the "free" parameter, ε^P, is in fact fixed at the level estimated on the historical time series of the variables of the equation: The "trivial" parameter, χ^P, is calculated residually such that the base year value of the endogenous variable, $X_{t_0}^P$, is exactly reproduced. In estimated equations of a macroeconometric model this will only occur if the stochastic regression residual is equal to zero for that year. The calibrated model therefore transfers a non-zero base year residual to the "trivial" parameter.

However, it should be noted that these differences is only a matter of traditions and not clear-cut, as macroeconometric modellers might of course also rely on the calibration procedure of CGE modellers for some equations (and in fact have occasionally done so), and CGE modellers might incorporate base year regression residuals as additive terms in their estimated equations (as it is in fact done in some CGE-models).

7.5 The benchmark data set

The calibration example of the preceeding section was based directly on the base year data set. However, as already mentioned, as we calibrate the stationary state solution of the model it is required that the chosen data set can be interpreted as depicting the stationary state of a model economy. There are several formal obstacles for regarding the base year data as mapping a stationary state. In addition, some variables has to be calculated on model consistent assumptions.

Table 7.3 and 7.4 lists all the variables of the model. Table 7.3 displays the variables which values are known from various sources, whereas table 7.4 displays the variables which values are calculated given the values of the know variables of table 7.3 and - if necessary - of the values of other calculated variables of table 7.4.

The sources of the known variables are indicated in the three parts of table 7.3. First, the values of some variables are measured directly by the external data described in appendix E. The amounts are measured in billions of Dkr. in the external data, i.e. the quantities are obtained by the price normalizations indicated in table 7.3, fixing some prices to 1 implying that the corresponding amounts in current prices and the quantities are equal. Finally, for some variables there is only one value consistent with stationary state. They are therefore fixed at those levels, even though it contradicts the actual data.

The values of the variables of table 7.4 are calculated on model consistent assumptions by using the model equations under the condition that a stationary state prevails. Most of these variables are calculated on a single equation basis. However, the values of the variables concerning the distribution of households' assets and consumption over generations are also the outcome of the calibration of the parameters of the households' intertemporal utility function which is done by solving several equations at the time.

The next four subsections deal with the more difficult parts of the construction of the benchmark data set. These are

- removing the formal obstacles for regarding the base data set as generated by a stationary economy

- compiling numbers for unobserved stock variables based on model-consistent assumptions.

- compiling numbers for unobserved installation costs based on model-consistent assumptions.

- performing joint calibration of the households' intertemporal parameters and calculation of the generational distribution of households' consumption and asset holdings.

1. Variables covered by actual data

C(TA)	Total private consumption
CATG(TA)	Net transfers from foreigners to the public sector
FA(TA)	Foreign assets
GA(TA)	Government debt
GC(TA)	Government consumption
GDP(TA)	Aggregate GDP at factor costs
H(ACTP,TA)	KL CES aggregate
I(ACTP,TA)	Gross investment (CES Armington index)
ID(ACTP,TA)	Domestic inputs to gross investments
IFO(ACTP,TA)	Foreign inputs to gross investments
LD(ACTP,TA)	Demand for labor
MP(ACTP,TA)	Input of private materials
MG(ACTP,TA)	Input of governmental materials
MPD(ACTP,TA)	Input of private domestic materials
MPF(ACTP,TA)	Input of private foreign materials
M(ACTP,TA)	Input of materials (CES Armington index)
TB(TA)	Trade balance
TRANSF(A,TA)	Age dependent public transfers to each female
TRANSFT(A,TA)	Age dependent public transfers to each female - taxed
TRANSM(AM,TA)	Age dependent public transfers to each male
TRANSMT(AM,TA)	Age dependent public transfers to each male - taxed
TRANSP(TA)	Public pensions to each pensioner
UB(TA)	Unemployment benefits rate
X(ACTP,TA)	Exports
Y(ACTP,TA)	Net output (gross output minus installation costs)

2. Normalized price variables [a]

PF(TA)	Foreign price
PH(ACTP,TA)	KL CES price index
PY(ACTP,TA)	Net output CES price index

3. Manipulated variables [b]

CA(TA)	Current account
GB(TA)	Government budget balance

Notes:

a) The unit wage costs, $(1+ta)W$, are also normalized to 1.

b) Net investments are only implicitly defined in the model. They are manipulated to their stationary value (zero) via the depreciation rate. The population figures, which are parameters, are also manipulated to stationary levels.

Table 7.3. Known variables

1. Variables calculated on a single equation basis	
A(TA)	Total household assets
CAT(TA)	Net transfers from foreigners to each adult
CO(TA)	Private consumption per very old person (CES Armington index)
COG(TA)	Consumption of the governmental good per very old person
COP(TA)	Consumption of the private good per very old person
COPD(TA)	Consumption of the dom. supp. private good per very old person
COPF(TA)	Consumption of the foreign supp. private good per very old person
DIV(ACTPP,TA)	Dividend from firm
DF(ACTPP,TA)	Debt of firm
HLUMP(TA)	Lump sum transfer from government to each adult
HY(A,TA)	Household non-interest income per adult
HYF(A,TA)	Household non-interest female income per adult
HYM(A,TA)	Household non-interest male income per adult
K(ACTP,TA)	Capital stock[a]
KB(ACTPP,TA)	Book value of capital stock
LF(A,TA)	Supply of female labour
LM(AM,TA)	Supply of male labour
MPK(ACTP,TA)	Marginal product of capital
PC(TA)	Consumer CES price index
PCP(TA)	Consumer CES price index for private good
PI(ACTP,TA)	Gross investments CES price index
PM(ACTP,TA)	Materials CES price index
PMG(ACTP,TA)	Governmental materials CES price index
PMP(ACTP,TA)	Private materials CES price index
Q(ACTPP,TA)	Shadow price of the marginal unit of capital
QB(ACTPP,TA)	Shadow price of the marginal unit of book capital
TAX(TA)	Tax revenue
V(ACTPP,TA)	Value of firm
W(TA)	Wage for one unit of labour
YTILDE(ACTP,TA)	Gross output
2. Variables calculated by the multi-equation intertemporal calibration	
ASSET(A,TA)	Households assets per adult-equivalent
C(A,TA)	Household consumption per adult-equivalent (CES Armington index)
CG(A,TA)	Household consumption of the governmental good per adult-equivalent
CL(A,TA)	Instantaneous utility per adult-equivalent
CP(A,TA)	Household consumption of the private good per adult-equivalent
CPD(A,TA)	Household consumption of the dom. supplied private good per adult-equivalent
CPF(A,TA)	Household consumption of the foreign supplied private good per adult-equivalent
Note:	a) The value of the capital stock valuated at replacement prices, $p^{iI}K^{i}$ $i=P,G$, is covered by actual data. The physical capital stock K^{i} is then inferred after the investment price, p^{iI}, is calculated.

Table 7.4. Calculated variables

7.5.1 *Removing formal obstacles for calibration of stationary state*

Inspecting the base year data set, it is seen that four formal obstacles for regarding the base year as a stationary state apply. The first concerns the changing population, while the last three concern a non-zero flow variable causing the corresponding stock variable to change contradicting the stationarity requirements.

1. The size and composition of the population at age and gender are changing

The fact that the population is not constant obviously prevents a stationary state from occuring. The actual base year population figures are therefore replaced with the stationary state population which emerges from extrapolating the number of births until the size and composition of the population settles, and the age profile reflects mortality. The extrapolated number of births is fixed at the level which results in a stationary state population at the same size as the base year population, but of course the composition of the population differs. The methodology is described in details in chapter 3.

2. There are non-zero net investments

The implication of non-zero net investments is that the capital stock changes during the base year counteracting a stationary state. One way to solve this problem is to define the physical rate of depreciation such that all base year gross investments consist of reinvestments (depreciations) leaving net investments equal to zero. This track is followed here, because it results in an estimate of the depreciation rate at 0.058 for the private sector being rather close to actual ratio of national accounts reinvestments to the capital stock at 0.053 in 1995. However, for the governmental sector the corresponding figures are 0.037 and 0.019 respectively. Of course, the usefulness of this method depends on actual net investments being small.[9]

Formally, from the physical capital accumulation identity (2.6) and the stationary state condition of a constant physical capital stock, $K_t = K_{t-1}$, we obtain the physical rate of depreciation of the capital stock, δ, as

$$\delta = \frac{I_{base}}{K_{base}} \tag{7.3}$$

where subindex *base* refers to actual base year figures ($t = base$). This implies that benchmark gross investments exactly equalize the actual base year figures as intended

$$I_{t_0} = \delta K_{t_0} = I_{base} \tag{7.4}$$

where $K_{t_0} = K_{base}$ (measured at the end of the base year). As usual, subindex t_0

[9] If this is not the case, it could alternatively be preferred to change the level of (gross) investments such that it coincides with the depreciation rate times the capital stock leaving net investments equal to zero. However changing the level of investments is more complicated as it changes several numbers of the input output table. In the creating the benchmark data set II, the manipulations of the rate of depreciations can be redone

refers to the benchmark figures ($t = t_0$) which are now allowed to deviate from the base year figures if necessary.

3. There is a public sector deficit

The public deficit implies that the size of the net public debt changes during the base year in conflict with the stationary state requirements. This is dealt with by simply defining a (negative) net lump sum transfer from the public sector to the households which balances the public budget.

The benchmark sum of lump sum transfers from the government to the households, Υ_{t_0}, is therefore defined by directly applying the government budget constraint (5.20). This assigns that value to Υ_{t_0} which corresponds to the ordinary budget surplus. Using (5.14) defining the relation between individual and aggregate public lump sum transfers, the benchmark public lump sum transfer to each adult is simply obtained as the average amount per adult

$$\tau_{t_0} = \frac{\Upsilon_{t_0}}{\sum_{b=18}^{77} N_{b,t_0}^{AF}} \tag{7.5}$$

This definition of public lump sum transfers to households has implications for the benchmark figures for aggregate household income. This is automatically dealt with because aggregate household income is defined in the benchmark data set by applying the models definition, cf. table 7.4.

This definition of benchmark lump sum transfers as a balancing item is also utilized to correct for some entries of the public account not being accurately represented. This is further described in appendix E which gives a complete representation of the public account in the model.

4. There is a current account surplus

The current account surplus leads to an increase in the stock of net foreign assets during the base year contradicting the requirement of stationarity. Again, the remedy is to define a suitable lump sum transfer which balances the current account.

The net transfer to each household from abroad, $\tau_{t_0}^W$, is defined such that the current account surplus is zero. The identity (5.15) gives the relation between the current account and the accumulation of net foreign assets. Noting that in stationary state the foreign debt is unchanged, i.e. $F_t = F_{t-1}$, (5.15) can be solved for the total lump sum transfers to households from abroad, TW_{t_0}, which gives a zero current account surplus

$$TW_{t_0} = -r_{t_0}F_{t_0} - TWG_{t_0} - \sum_{i=P,G} p^i_{t_0}\left(1+t^{iX}_{t_0}\right)X^i_{t_0} \tag{7.6}$$

$$+\frac{1}{1+t^t_{t_0}}\left(C^{PF}_{t_0} + \sum_{i=P,G}\left(M^{iPF}_{t_0} + I^{iF}_{t_0}\right)\right) \tag{7.7}$$

The total lump sum transfers to households from abroad, TW_{t_0}, consists of the sum of the individual transfers, $\tau^W_{t_0}$, as defined in (5.13). Inverting this equation, the foreign lump sum transfers to each adult, $\tau^W_{t_0}$, is obtained as the average amount per adult

$$\tau^W_{t_0} = \frac{TW_{t_0}}{\sum_{b=18}^{77} N^{AF}_{b,t_0}}$$

Also, this definition of benchmark transfers from abroad has implications for the benchmark figures for aggregate household income. Again, this is automatically dealt with because aggregate household income is simply defined in the benchmark data set by applying the models definition, cf. table 7.4.

This definition of benchmark lump sum transfers as a balancing item is also utilized to correct for some entries of the current account not being accurately represented. This is further described in appendix E which gives a complete representation of the current account in the model.

7.5.2 Compiling numbers for unobserved stock variables

There are three stock variables in the model which are not covered by the base year data set. They are all observable, but the existing economic statistics only supply partial evidence not covering the model's theoretical concepts adequately. An important reason is that not all firms are organized as corporations as assumed in the model.

1. The debt of the representative firm.

The corporate debt rule (2.5) gives the relation between the stock of corporate debt, $B^c_{t_0}$, and the value of the capital stock $p^{PI}_{t_0}K^P_{t_0}$

$$B^c_{t_0} = gp^{PI}_{t_0}K^P_{t_0} \tag{7.8}$$

From this the base year corporate debt can be calculated given an assumption about the corporate debt share, g. As seen in table 7.1 the share is fixed to 60 per cent, cf. appendix E.

2. The book value of the capital stock.

The benchmark book value of the capital stock, $\hat{K}_{t_0}$, can be calculated using the book capital accumulation identity (2.7) and the stationary state condition of a constant book value of the capital stock, $\hat{K}_t = \hat{K}_{t-1}$, as

$$\hat{K}_{t_0} = \frac{p^{PI}_{t_0} I^P_{t_0}}{\hat{\delta}_{t_0}} \tag{7.9}$$

3. The value of the representative firm.

In principle, the value of the representative firm corresponds to a stock market value concept. However, far from all firms are organized as corporations making it difficult to measure their market value directly. Anyway, the stationarity requirement imposes a link between the flow of dividends and the value of the firms. We therefore start by measuring benchmark dividends by applying the definition of dividends (2.4) imposing the stationary state conditions $B^c_t = B^c_{t-1}$ and $\hat{K}_t = \hat{K}_{t-1}$. This leads to the benchmark dividends

$$\begin{aligned} D_{t_0} &= \left(1 - t^c_{t_0}\right)\left(p^P_{t_0} Y^P_{t_0} - p^{PM}_{t_0} M^P_{t_0} - \left(1 + t^a_{t_0}\right) W_{t_0} L^P_{t_0} - r_{t_0} B^c_{t_0}\right) \\ &\quad - p^{PI}_{t_0} I^P_{t_0} + t^c_{t_0} \hat{\delta}_{t_0} \hat{K}_{t_0} \end{aligned} \tag{7.10}$$

This is inserted in the arbitrage condition (2.1) giving the relation between dividends and the value of the firm. In addition the stationary state requirement $V_t = V_{t-1}$ is imposed, giving

$$V_{t_0} = \frac{1 - t^d_{t_0}}{1 - t^r_{t_0}} \frac{D_{t_0}}{r_{t_0}} \tag{7.11}$$

This procedure of measuring a stock from the corresponding flow in a given year may be problematic, because the flows can vary much from year to year. It is however preferred instead of imputing the value of firms inclusive the large number of non-corporations from various incompatible sources.

7.5.3 Compiling numbers for installation costs

Installation costs are not observable. They are therefore imputed by using the model, i.e. by assuming that installation costs in the benchmark data set amounts to exactly that value which is compatible with a stationary state. Installation costs are assumed to be zero for the public sector.

The starting point is that the model gives a relationship between the size of the capital stock, the gross surplus (the value of gross output minus the value of all inputs except capital) and the marginal value product of capital. The idea is to compare the model calculated gross surplus with the gross operating surplus (in Danish "bruttorestindkomsten") as defined in the national accounts. The difference is defined as the installation costs. From this the parameters in the installation cost function are inferred.

The gross production function for the private sector $\tilde{Y}_t^P = F^P\left(M_t^P, K_{t-1}^P, L_t^P\right)$ exhibits constant returns to scale, i.e. Eulers theorem yields

$$\begin{aligned} F^P\left(M_t^P, K_{t-1}^P, L_t^P\right) &= \frac{\partial F^P}{\partial M^P}\left(M_t^P, K_{t-1}^P, L_t^P\right) M_t^P + \\ &\quad \frac{\partial F^P}{\partial K^P}\left(M_t^P, K_{t-1}^P, L_t^P\right) K_{t-1}^P + \\ &\quad \frac{\partial F^P}{\partial L^P}\left(M_t^P, K_{t-1}^P, L_t^P\right) L_t^P \end{aligned}$$

Utilizing the first order conditions for the demand for labor (2.17) and the demand for materials (2.18) of the representative firm and the expression (6.9) for the marginal product of capital in stationary state we obtain

$$K_{t_0}^P = \frac{p_{t_0}^P \tilde{Y}_{t_0}^P - \left(p_{t_0}^{PM} M_{t_0}^P + \left(1 + t_{t_0}^a\right) W_{t_0}\right)}{p_{t_0}^P {\beta^P}^{SS}(p^P)} \tag{7.12}$$

for the benchmark values of the variables (indicated by subindex t_0). The numerator of (**??**) defines what we termed as the gross surplus above. Using (2.8) which defines the relation between the gross output, the installation costs and the net output, the gross surplus can be written as

$$p_{t_0}^P \tilde{Y}_{t_0}^P - \left(p_{t_0}^{PM} M_{t_0}^P + \left(1 + t_{t_0}^a\right) W_{t_0}\right) = \Pi_{NA,t_0}^P + p_{t_0}^P \Phi^P(I_{t_0}^P, K_{t_0}^P) \tag{7.13}$$

where Π_{NA,t_0}^P is equal to the gross operating surplus as defined in the national accounts and $\Phi^P(I_{t_0}^P, K_{t_0}^P)$ as usual denotes the real installation costs.

The stationary state capital stock consistent with the benchmark data is determined by (7.12). If there were no installation costs, the numerator of (7.12) would be equal to Π^P_{NA,t_0}. The denominator is equal to the benchmark marginal value of capital. Then (7.12) determines the benchmark capital stock, $K^P_{t_0}$. However, we already have an observation of $K^P_{t_0}$ from the national accounts. We then turn (7.12)-(7.13) around and use the observation of $K^P_{t_0}$ and Π^P_{NA,t_0} and the calibrated value of $\beta^{P\,SS}(p^P)$ to determine $\Phi^P(I^P_{t_0}, K^P_{t_0})$.

Solving for $\Phi^P(I^P_{t_0}, K^P_{t_0})$ and utilizing the calibration convention $p^P_{t_0} = 1$, the benchmark installation costs compatible with the benchmark (= base year) capital stock, $K^P_{t_0}$, and the benchmark (= base year) national account gross operating surplus, Π^P_{NA,t_0}, are found as

$$\Phi^P(I^P_{t_0}, K^P_{t_0}) = \beta^{P\,SS}(p^P)K^P_{t_0} - \Pi^P_{NA,t_0} \tag{7.14}$$

Inserting the expression for Φ^P given by (2.13) and applying the standard value $\iota = 1$, gives

$$\phi^P \frac{\left(I^P_{t_0}\right)^2}{K^P_{t_0}} = \beta^{P\,SS}(p^P)K^P_{t_0} - \Pi^P_{NA,t_0} \tag{7.15}$$

Taking into consideration the benchmark definition of δ^P, (**??**), ϕ^P is obtained as

$$\phi^P = \frac{\beta^{P\,SS}(p^P)K^P_{t_0} - \Pi^P_{NA,t_0}}{(\delta^P)^2 K^P_{t_0}}$$

The stationary state marginal product of capital, $\beta^{P\,SS}(p^P)$, is given by (6.9) which shows that $\beta^{P\,SS}(p^P)$ depends on ϕ^P. Inserting this expression for $\beta^{P\,SS}(p^P)$ and reducing, we obtain (subindices t_0 are suppressed)

$$\phi^P = \frac{\frac{p^{PI}(p^P,1)}{p^P}\left[r\left(g + \frac{(1-t^r)}{(1-t^g)(1-t^c)}(1-g)\right) + \delta^P + \frac{(1-t^r)rt^c\left(\delta^P - \hat{\delta}\right)}{(1-t^c)\left((1-t^r)r + (1-t^g)\hat{\delta}\right)}\right] - \frac{\Pi^P_{NA}}{K^P}}{2\delta^P\left(\delta^P - \left(\frac{(1-t^r)}{(1-t^g)}r + \delta^P\right)\right)}$$

This gives the value of ϕ^P which - given $\iota = 1$ - delivers the level of benchmark installation costs consistent with the benchmark physical capital stock, the physical rate of depreciation (and therefore the benchmark level of (re)investments) and the benchmark gross operating surplus.

In accouting terms, the installation costs are entered into the input output table as both an additional primary input and as an intermediate demand, cf. the shaded cells of table 7.5, which indicates how the base year input output table, cf. table 1 of appendix E, is expanded. This assures that the accounting identities continue to apply. The private sector gross output at basic values is enhanced by installation costs amounting to 44.633 billion Dkr. or 3.6 per cent of the original national accounts level excluding installation costs and 30.7 per cent of total private gross investments. The reader is recommended to confer appendix E for further comments on the input output data.

	Domestic production		Consumption		Gross investments		Exports	Inst.	Sum
	Private	Public	Private	Public	Private	Public		costs	
1.a. Domestic private production	437793	59986	312254	0	106769	14039	311187	44633	1286661
1.b. Domestic public production	7653	5798	18928	243905	0	0	216	0	276500
2.a. Imports of private production	143293	8804	85205	0	28756	1451	0	0	267509
2.b. Imports of public production	0	0	0	0	0	0	0	0	0
3.a. Quantity commodity taxes	0	0	29500	0	0	0	0	0	29500
3.b. Ad valorem commodity taxes, net	7526	18081	73235	0	9630	4011	-4433	0	108050
4. Uses at market prices (1-3)	596265	92669	519122	243905	145155	19501	306970	44632.74	1968219.7
5. Compensation of employees	341383	173776							
6. Gross operating surplus	304380	10055							
7. Installation costs	44633	0							
8. Gross output at basic values (4-7)	1286661	276500							

Table 7.5. The benchmark input output table with installation costs, mill. Dkr.

7.5.4 *Joint calibration of the intertemporal utility function and compilation of some household benchmark data*

The intertemporal utility function contains the three parameters the rate of time preference, θ, the intertemporal elasticity of substitution, S, and the bequest parameter, ξ. The calibration of these three parameters are performed under the two additional restrictions that households in the aggregate display exactly the macro level of households' private consumption and possession of assets in the benchmark data set. These two additional constraints determine two of the three parameters. The remaining parameter must be fixed in advance.

Formally, the calibration is the solution to the following equations: the budget constraint, (4.15), the Keynes Ramsey rule, (4.37), the optimal Keynes Ramsey like bequest rule, (**??**), the definition of instantanous utility, (4.10), and the two already mentioned additional constraints. This gives the block of stationary state equations for the calibration year

$$a_{b,t_0} = (1 + \tilde{r}_{b,t_0})\, a_{b-1,t_0} + \tilde{y}_{b,t_0} - P_{t_0} Q_{b,t_0} \ , \ 18 \leq b \leq 78 \tag{7.16}$$

$$\frac{Q_{b+1,t_0}}{Q_{b,t_0}} = \left(\frac{1 + \tilde{r}_{b+1,t_0}}{1+\theta} \right)^S \ , \ 18 \leq b < 77 \tag{7.17}$$

$$Q_{78,t_0} = \left(\xi \frac{1 + \tilde{r}_{78,t_0}}{1+\theta} \right)^S Q_{77,t_0} \tag{7.18}$$

$$Q_{b,t_0} = C_{b,t_0} - Z_{b,t_0} \ , \ 18 \leq b \leq 77 \tag{7.19}$$

$$\sum_{b=18}^{77} N^{EF}_{b,t_0} C_{b,t_0} = C_{t_0} - N^O_{t_0} C_{o,t_0} \tag{7.20}$$

$$\sum_{b=18}^{77} N^{EF}_{b,t_0} a_{b,t_0} = A_{t_0} \tag{7.21}$$

(7.16)-(7.19) are part of the ordinary stationary state model, while (7.20) and (7.21) provide the additional restrictions. C_{t_0} is total private consumption in the economy and $N_t^O C_{o,t}$ equals total consumption of the very old persons (who have survived the planning horizon of the ordinary households), i.e. the right hand side of (7.20) gives the total private consumption available for the ordinary households. (7.21) only reproduces the definition of total household assets, A_{t_0}, cf. (5.9).

a_{17,t_0}, a_{78,t_0}, $\tilde{r}_{b,t_0}$, $\tilde{y}_{b,t_0}$, P_{t_0}, Z_{b,t_0} and C_{o,t_0} are treated as exogenous to the problem, because in the entire stationary state model these variables are either exogenous or determined endogenously in equations other than (7.16)-(7.19). The non-interest income per adult-equivalent corrected for the disutility value of work, $\tilde{y}_{b,t_0}$, depends on inheritage plus other income sources, cf. (4.17), and (4.3) defining the non-interest income per adult of a household of age b, y_{b,t_0}. This inheritage is related to the bequest of the eldest households, which is again related to their asset holdings, a_{77,t_0}, just before they die. The exogenity of $\tilde{y}_{b,t_0}$ therefore demands that a_{77,t_0} is know.[10]

C_{t_0}, A_{t_0} are also exogenous which is the essence of the procedure. N^{EF}_{b,t_0} and N^O_t are a parameters. a_{b,t_0} ($b \neq 17$, $b \neq 78$), Q_{b,t_0}, C_{b,t_0} are treated as endogenous to the problem. This leaves 183 equations (61+59+1+60+1+1) to determine 181 variables (60+61+60). This means that two of the three intertemporal parameters can be determined from the system (7.16)-(7.21), leaving one degree of freedom to fix one parameter exogenously. The two parameters are in fact determined by the additional restrictions (7.20) and (7.21). We choose to fix the rate of time preference, θ, and calibrate the intertemporal elasticity of substitution, S, and the bequest parameter, ξ, from (7.16)-(7.21), but basically it does not matter, because the same calibration can of course be obtained by alternatively fixing one of the other parameters at the value obtained in the first calibration.

Although the exogenous C_{t_0} and A_{t_0} are purely aggregate data, it should be stressed that the calibration (i.e. the fixing of θ and a_{77,t_0}) is performed such that the resulting model is roughly able to replicate the typical picture of the development of households assets revealed from the existing micro data: The households' net wealth grows over most of the life cycle, but then declines somewhat several years before they die, still leaving a considerable bequest. Presently, it is assumed that 4 per cent of the total assets holdnings of all the households are owned by the housholds which are 77 years old, i.e. the bequest of housholds corresponds to 4 per cent of all their assets.

Table 7.2 shows that the preference for leaving bequest is calibrated to 0.603. The interpretation is that the household equates the utility of leaving 1 Dkr. of bequest with 0.603 Dkr. of consumption at the last period of its life, i.e. with roughly three times last year's consumption.

7.6 Calibration on the benchmark data set

7.6.1 Net values, excise tax rates and market prices

[10] The assumed value of a_{77,t_0} is exactly reproduced endogenously by the solution to the intertemporal calibration problem.

The indirect tax rates applied to materials, $t_{t_0}^{iM}$, $i = P, G$, and investments, $t_{t_0}^{iI}$, $i = P, G$, are calibrated analogous to the calibration of the export tax rate, $t_{t_0}^{iX}$.

The relation between the tax base, the tax rates and the total tax revenue is given in (5.33) for excise taxes applied to material inputs. Defining the separate revenue from the private sector, $MT_{t_0}^{P}$, and from the governmental sector, $MT_{t_0}^{G}$, the total materials tax revenue can also be written

$$MT_{t_0} = MT_{t_0}^{P} + MT_{t_0}^{G}$$

where

$$MT_{t_0}^{i} = t_{t_0}^{iM}\left(p_{t_0}^{P} M_{t_0}^{iPD} + M_{t_0}^{iPF} + p_{t_0}^{G} M_{t_0}^{iG}\right) , \qquad i = P, G$$

Solving these for the excise tax rates give the equations

$$t_{t_0}^{iM} = \frac{MT_{t_0}^{i}}{p_{t_0}^{P} M_{t_0}^{iPD} + M_{t_0}^{iPF} + p_{t_0}^{G} M_{t_0}^{iG}} , \qquad i = P, G \tag{7.22}$$

where the numerator is the revenue and the denominator is the tax base of the tax in question. In (7.22) all variables on the right hand sides are obtained directly from the input output table and the Harberger normalizing convention for the net price, $p_{t_0}^{i} = 1$. The market prices are then calculated by simply multiplying the net prices with one plus the relevant excise tax rate. For example, the market price of privately domestically produced materials is equal to $\left(1 + t_t^{iM}\right) p_t^{P}$ for inputs in sector i.

Similarly, (5.34) gives the relation between the tax base, the tax rates and the total tax revenue for excise taxes applied to investments. Defining the separate revenue from private investments, $IT_{t_0}^{P}$, and governmental investmnets, $IT_{t_0}^{G}$, the excise taxes on investments for the two materials tax rates are found as

$$t_{t_0}^{iI} = \frac{IT_{t_0}^{i}}{p_t^{P} I_t^{iD} + I_t^{iF}} , \qquad i = P, G$$

where $IT_{t_0}^{P}$ is the revenue from the private sector and $IT_{t_0}^{G}$ is the revenue from the governmental sector.

For simplicity it is assumed that the quantity excise tax rates of private consumption, t_t^{CP} and t_t^{CG}, are equal. Therefore they can be calibrated by solving the revenue equation (5.28) for this uniform tax rate, giving

$$t_{t_0}^{CP} = t_{t_0}^{CG} = \frac{RC_{t_0}}{C_{t_0}^{P} + C_{t_0}^{G}}$$

The base year value added tax rate, $t_{t_0}^{vat}$, is calculated by inverting the revenue equation (5.27) giving

$$t_{t_0}^{VAT} = \frac{VAT_{t_0}}{\left(p_{t_0}^{CP} + t_{t_0}^{CP}\right) C_{t_0}^{P} + \left(p_{t_0}^{G} + t_{t_0}^{CG}\right) C_{t_0}^{G}}$$

Although the VAT rate in 1995 was 0.25, the calibrated value of $t_{t_0}^{VAT}$ is only 0.164, cf. table 7.2. The main explanation is that gross rents (consumption of the services of dwellings) are almost entirely exempted from excise duties.

The toll rate, $t_{t_0}^{t}$, is calibrated by solving (5.36) for the toll rate, giving

$$t_{t_0}^{t} = \frac{TTB_{t_0}}{1 - TTB_{t_0}}, \qquad TTB_{t_0} = \frac{TT_{t_0}}{C_{t_0}^{PF} + \sum_{i=P,G}\left(M_{t_0}^{iPF} + I_{t_0}^{iF}\right)}$$

7.6.2 The production function and the atemporal utility function

The production function is specified as a nested CES-function and the atemporal nests of the utility function are also CES-functions. The calibration of these CES-functions can be illuminated by considering an example: the private materials nest in the nested CES function of producers, where the producer in question combines privately produced materials of domestic and foreign origin to obtain the intermediate product privately produced materials. This CES-function which is defined for both the private producers ($i = P$) and the governmental producers ($i = G$) writes

$$M_t^{iP} = \left[\mu_{iYMPD}\left(M_t^{iPD}\right)^{\frac{\sigma_{iYMP}-1}{\sigma_{iYMP}}} + \mu_{iYMPF}\left(M_t^{iPF}\right)^{\frac{\sigma_{iYMP}-1}{\sigma_{iYMP}}}\right]^{\frac{\sigma_{iYMP}}{\sigma_{iYMP}-1}}, \; i{=}P,G$$

cf. (2.12). Appendix B shows that for each set of equations ($i = P$ and $i = G$ respectively) the two distribution parameters can be calibrated as

$$\mu_{iYMPD} = \frac{(1+t_{t_0}^{iM})p_{t_0}^{P}}{p_{t_0}^{iM}} \left(\frac{M_{t_0}^{iPD}}{\frac{(1+t_{t_0}^{iM})p_{t_0}^{P} M_{t_0}^{iPD} + (1+t_{t_0}^{iM}) M_{t_0}^{iPF}}{p_{t_0}^{iM}}} \right)^{\frac{1}{\sigma_{iYMP}}}, \; i{=}P,G \tag{7.23}$$

$$\mu_{iYMPF} = \frac{(1+t_{t_0}^{iM})}{p_{t0}^{iM}} \left(\frac{M_{t_0}^{iPF}}{\frac{(1+t_{t_0}^{iM})p_{t_0}^{P} M_{t_0}^{iPD} + (1+t_{t_0}^{iM}) M_{t_0}^{iPF}}{p_{t_0}^{iM}}} \right)^{\frac{1}{\sigma_{iYMP}}}, \; i{=}P,G \tag{7.24}$$

where the CES price index of the aggregate of domestic and imported inputs, $p_{t_0}^{iM}$, is normalized as

$$p_{t_0}^{iM} = 1 + t_{t_0}^{iM},\ i{=}P, G \tag{7.25}$$

In appendix B (remark 1) it is shown that one is free to normalize the CES output price index to any convenient positive value. The calibration formula automatically assures that the CES output quantity index is implicitly rescaled accordingly (via the calibration of the distribution parameters), such that the value of output is unchanged. Of course, one is also free to normalize the input prices to any convenient positive value, as long as input quantities are defined correspondingly such that the values are unaffected.

As should also be clear from the precceding subsection on the calibration of excise tax rates, it is simply assumed that for each production sector, the sector's materials input excise tax rate, $t_{t_0}^{iM}$, is applied equally to the domestically produced materials, $M_{t_0}^{iPD}$, and to the imported materials, $M_{t_0}^{iPF}$. In agreement with this, the net price of the former is $p_{t_0}^{P} = 1$, and the net price of latter is implicitly assumed to be equal to one (it is the numeraire and therefore it is always assumed to be equal to one). Accordingly, the corresponding market prices of the two inputs of sector i are both equal to $1{+}t_{t_0}^{iM}$. In accordance with the price normalization conventions, the aggregate CES (market) price index of sector i, $p_{t_0}^{iM}$, is also normalized to $1 + t_{t_0}^{iM}$, cf. (7.25), as this implies that the net price of the aggregate privately produced materials input is equal to one.

It is required that the parameters are calibrated such that the actual base year material inputs of sector i, $M_{t_0}^{iPD}$ and $M_{t_0}^{iPF}$, are exactly reproduced by the model. For each sector i, this implies the two restrictions (7.23) and (7.24) on the value of the three CES-parameters: The two distribution parameters, μ_{iYMPD} and μ_{iYMPF}, and the elasticity of substitution, σ_{iYMP}, - given the base year value of the two inputs, $M_{t_0}^{iPD}$ and $M_{t_0}^{iPF}$, the materials excise tax rate calibrated in (7.22), the price normalizing conventions fixing $p_{t_0}^{P}$ (and implicitly also the numeraire import price) at one and the convention (7.25). With two restrictions and three parameters, there is one degree of freedom. This is utilized to set $\sigma_{PYMP} = 1.1$, $\sigma_{GYMP} = 1.2$, cf. table 7.1, which for both sectors correspond to the targeted import price elasticity of raw materials at -0.9, cf. appendix E. The different elasticities of substitution for the two sectors result imply the same own price elasticity do to differences in the cost shares of the inputs.

The calibrated values of the distribution parameters are displayed in table 7.2. All CES functions are calibrated in the same way. The calibration of the capital labor nest of the nested CES production function of the firm, requires that the price of capital is identified. This is done by inferring it from the marginal product of capital in stationary state. The CES production function of the capital labor nest of the private sector, (2.10), is repeated here

$$H_t^P = \left[\mu_{PYHK}\left(K_{t-1}^P\right)^{\frac{\sigma_{PYH}-1}{\sigma_{PYH}}} + \mu_{PYHL}\left(L_t^P\right)^{\frac{\sigma_{PYH}-1}{\sigma_{PYH}}}\right]^{\frac{\sigma_{PYH}}{\sigma_{PYH}-1}}$$

Appendix B.1 (remark 1) shows that the two distribution parameters can be calibrated as

$$\mu_{PYHK} = \frac{p_{t_0}^{PH}\beta^{P\,SS}(p^P)}{p_{t_0}^{PiH}}\left(\frac{K_{t_0}^P}{\frac{\beta^{P\,SS}(p^P)K_{t_0}^P+(1+t_{t_0}^a)W_{t_0}L_{t_0}^P}{p_{t_0}^{PH}}}\right)^{\frac{1}{\sigma_{PYH}}} \tag{7.26}$$

$$\mu_{PYHL} = \frac{p_{t_0}^{PH}(1+t_{t_0}^a)W_{t_0}}{p_{t_0}^{PH}}\left(\frac{L_{t_0}^P}{\frac{\beta^{P\,SS}(p^P)K_{t_0}^P+(1+t_{t_0}^a)W_{t_0}L_{t_0}^P}{p_{t_0}^{PH}}}\right)^{\frac{1}{\sigma_{PYH}}} \tag{7.27}$$

where $\beta^{P\,SS}(p^P)$ is the stationary state marginal product of capital (6.9) and $p_{t_0}^{PH}$ is the CES price index corresponding to the CES capital labor aggregate index, $H_{t_0}^P$. In stationary state, the price of capital must equal its marginal value product, i.e. it must be equal to $p_{t_0}^{PH}\beta^{P\,SS}(p^P) = \beta^{P\,SS}(p^P)$ given the that $p_{t_0}^{PH}$ is conveniently normalized to one.[11] $\beta^{P\,SS}(p^P)$ can be calibrated on the benchmark data and parameters calibrated elsewhere, cf. (6.9). Given the assumed elasticity of substitution between capital and labor, $\sigma_{PYH} = 0.6$, cf. table 7.1, the two distribution parameters, μ_{PYHK} and μ_{PYHL}, are then calibrated to the values in table 7.2 using the above formulas.

The governmental sector is assumed to posses a politically determined capital stock which is not equal to the cost minimizing capital stock. In the calibration of the capital labor nest of the governmental sector it is simply assumed that the stationary state marginal product of capital of this sector, $\beta^{G\,SS}(p^G)$, is equal to the ratio of the base year national account gross operating surplus to the capital stock. In the

[11] The calibration formula assures that the CES output price index ($p_{t_0}^{PH}$) can be fixed to any convenient value, cf. appendix B.

governmental sector analogue to (7.14) we set the installation costs to zero giving the calibrated value

$$\beta^{G\,SS}(p^G) = \frac{\Pi^G_{NA,t_0}}{K^G_{t_0}} \tag{7.28}$$

Of course, $\beta^{G\,SS}(p^G)$ deviates from $\beta^{P\,SS}(p^P)$. Given this calibrated value of $\beta^{G\,SS}(p^G)$, the calibration of the governmental CES-functions proceeds as for the private producers.

7.6.3 Labor demand, labor supply, maximum working time and underemployment

The units of measurements are defined such that base year unit wage costs, $(1 + t^a_{t_0})W_{t_0}$, are normalized to one. This implies that employment is measured in units of the base year's unit wage costs. Accordingly, total base year employment, $L_{t_0} = L^s_{t_0}$, is defined as the base year wage sum in billions of Dkr.

The aggregate labor supply, L^s_t, is defined as the product of the individual (identical) labor supplies, ℓ_t, and the number of individuals belonging to the labor force, N^W_t, cf. (5.6). As labor supply equals demand, individual employment is obtained from this as the simple average

$$\ell_{t_0} = \frac{L^s_{t_0}}{N^W_{t_0}} \tag{7.29}$$

Inserting in the labor supply function (4.26), repeated here

$$\ell_{t_0} = \left(\frac{\left[W_{t_0} - T^w_{t_0}\left(W_{t_0}\right)\right] - \left[b_{t_0} - T^b_{t_0}\left(b_{t_0}\right)\right]}{\gamma_1 P_{t_0}} \right)^{\gamma} \tag{7.30}$$

the scale parameter of this function, γ_1, can be found as

$$\gamma_1 = \left(\left[W_{t_0} - T^w_{t_0}\left(W_{t_0}\right)\right] - \left[b_{t_0} - T^b_{t_0}\left(b_{t_0}\right)\right]\right) \ell_{t_0}^{-\frac{1}{\gamma}} P_{t_0}^{-1} \tag{7.31}$$

cf. table 7.2, for given values of the base year data and the labor supply elasticity, $\gamma = 0.1$, cf. table 7.1. The aggregate rate of unemployment is defined as

$$\overline{U}_{t_0} = \frac{\overline{\ell} - \ell_{t_0}}{\overline{\ell}} \frac{N^W_{t_0}}{N^W_{t_0}} \tag{7.32}$$

from which the maximum working time measured in the base year's unit wage costs, $\overline{\ell}$, is inferred as

$$\overline{\ell} = \frac{\ell_{t_0}}{1 - \overline{U}_{t_0}} \tag{7.33}$$

Tabel 7.2 shows that $\overline{\ell}$ =0.209466 wage costs units of the base year. This is because ℓ_{t_0} = 0.187472 million Dkr. measured in the wage costs units of the base year (=1) and $\overline{U}_{t_0}$ =0.105.

7.7 Solution of the computer-version of the model

The computer program of the model is coded in the GAMS language (Brooke, Kendrick and Meeraus; 1988). GAMS allows the user to call a number of different solution algorithms for the numerical simulation. The present model is solved using the CONOPT2 solver. See Drud (1985, 1997).

The model is solved in the following way. First, a stationary state version is solved. The stationary state version is obtained by simply deleting all reference to time in the dynamic model. Then the dynamic model is solved, using the stationary state solutions as terminal conditions for the variables $a_{i,t}$, V_t, λ_{1t} and λ_{2t}. For these variables the model is solved for $t = 0, 1, 2, ..., T$, but for the remaining variables the model is only solved for $t = 1, 2, ..., T$,where T is a number large enough to allow the model to reach the new stationary state. If all predetermined variables are assigned their stationary state values, the dynamic model will be in the stationary state for all periods. The dynamic model is solved by stacking the equations for all periods, so a simulation with the dynamic model for T periods roughly gives T times the number of equations as in the stationary state version of the model.

It is an obvious choice to fix the import price in Dkr. inclusive custom and import duties as exogenously. It is convenient that it also operates as a numeraire. Due to Walras' law, it is necessary to fix one absolute price as a numeraire. The choice has no implications for the solution as the model is only able to determine relative prices.

As mentioned in section 3.1, the programming of the model enables the model-user to specify the number of generations in the model and thereby the length of periods in the dynamic model. The cost of specifying a short period, i.e. a high frequency of t, is that the number of periods before the model converges to the stationary state increase accordingly. The computational burden is, as explained above, increasing proportionally with the number of periods. For convenience the model is therefore calibrated to t having a frequency of 5 years. It should be remeinded that compared to t having an annual frequency this changes the relative size of stock and flow variables as a stock variable still represents an item at a certain point in time (end of the period), whereas the flow variable now represents a flow being 5 times larger. Also, interest and discount rates should be inflated accordingly.

Chapter 8
DYNAMIC EFFECTS OF POLICY EXPERIMENTS

In this section we perform X experiments in order to reveal how different parts of the model react to changes in policy variables. The first Y of the experiments focus on the impacts of changes in various tax rates influencing the behavior of households and firms in different ways. Two of the experiments consist of a reduction in the average pensioner age and a reduction of fertility. Finally, the last two experiments illustrate the effects of an increase in the public expenditure and of an increase in the age dependent transfers.

All experiments are conducted as unforeseen changes implemented at the beginning of period 1 (the end-of-period dating rule is described in section 1.4). For period $t = 0$ the variables assume their value in the steady state baseline. Correspondingly predetermined stock variables for $t = 0$ (ultimo period 0) are not altered by the shocks, while forward-looking variables such as firm value can jump at period zero. It should be stressed that the public sector budget is balanced in each period *via* the lump sum transfers to households, so that for example tax rate cuts are counteracted by increased lump sum taxes (decreased lump sum transfers) of households.

For each experiment a flow chart - that indicates the main channels through which the policy change in question affects the dynamics of the macroeconomic equilibrium - will be presented. Further 12 standard figure sets (with a variable number of diagrams in each set) are produced from the automatic reporting program for each of the experiments. The 12 standard figure sets consists of diagrams showing variables in the following categories

1. Output
2. Value-added and factor-intensities
3. Labor input and wage

4. Capital stock and investments
5. Input of materials
6. Consumption variables
7. Foreign trade variables
8. Savings and assets
9. Decomposition of firm variables
10. Decomposition of real human capital
11. Decomposition of tax revenues
12. Prices

These diagrams show the quantitative effects of the experiments through time. There are three kinds of diagrams. The first type displays for the variable in question, the variable's value in the counterfactual divided by the value in the baseline times 100[1]. The second type of diagrams decompose this index value showing the main contributions to the result, and the third outlines the effects on different generations. When the x-axis measures time, each unit represents 5-year periods as the model is solved for 5-year periods. Correspondingly, when the x-axis measures age, each unit represents 5-year age intervals. In the plots where the x-axis measures time, the steady state solution is indicated by an "−".

According to relevance only a selection of diagrams (varying across experiments) from the total figure set will be shown in connection with the analysis of the experiments. The interpretation of the diagrams will be explained in detail under the discussion of the first experiment.

8.1 Labor income tax rate reduced by 5 percentage points

The experiment consists of a reduction in the (marginal) tax rate on wage income by 5 percentage points. The reduction is financed through a decrease in the lump

[1] An exception is the present value of utility which is a negative number, i.e. a numerical decline marks an improvement. For utility is therefore shown 100 times the baseline value divided by the counterfactual value, because changes in this index reflects changes in utility properly.

sum transfer to households from the government. Wage income is here defined solely as income from employment. Since pensions and other age-specific taxable transfers are indexed to the pre-tax wage rate (net of labor market contributions) they do not benefit from the wage tax cut. Thus the policy implies a shift in the tax burden from the generations who are active in the labor force - i.e. generations between 18 and 60 years of age - to the pensioners.

There are two main channels through which the reduction in the labor income tax rate affects the economy. First, the reduction in the marginal tax rate implies a permanent outward shift in the labor supply curve. This generates an increase in the marginal product of capital which leads to increased investments and gradually increased capital stock. Second, the fact that the experiment favors the younger generations who are in the labor force at the expense of the elder generations, affects the life cycle behavior of the agents. There are two effects that counteracts one another: Young agents increase their propensity to save so that they are able to smooth consumption over the life cycle and second the reduction in the marginal tax rate implies that the activity in the economy is increased which adds to the value of firms and gradually also to the level of human capital in the economy. The two latter effects tend to increase consumption whereas the former tends to reduce consumption. The result of the simulation reveals that in the first period the net effect on consumption is practically zero, whereas the expansive effects dominates in all following periods.

It is not too surprising that reducing the distortionary tax on labor supply and replacing it with a non-distortionary lump sum tax leads to an expansion of the economy. However, the simulation reveals that this does not lead to a Pareto improvement, as the generations, who are in the last half of their life cycle when the policy shock hits the economy, will experience a deterioration of their utility. On the other hand young generations currently alive and all future generations will gain from the reduction in the marginal tax rate on labor income.

	Initial stationary state	5 years	10 years	25 years	50 years	New stationary state
Real private consumption	416	423	428	435	439	439
		(1.5)	(2.7)	(4.5)	(5.4)	(5.5)
Real GDP (at factor prices)	830	834	839	848	852	853
		(0.6)	(1.2)	(2.2)	(2.7)	(2.8)
Employment, index	100.0	103.1	103.2	103.3	103.3	103.3
		(3.1)	(3.2)	(3.3)	(3.3)	(3.3)
Capital stock	2755	2811	2827	2837	2840	2840
		(2.0)	(2.6)	(3.0)	(3.1)	(3.1)
Real value of firms	1380	1420	1422	1423	1423	1423
		(2.9)	(3.0)	(3.1)	(3.1)	(3.1)
Foreign assets	-266	-231	-194	-126	-93	-86

Note: The numbers in parantheses are the percentage change compared to the initial stationary state

Table 8.1. Dynamic adjustment in the labor income tax experiment

Table 1 shows the dynamic evolution of the main macroeconomic variables. The speed of adjustment differs significantly between production variables and consumption.

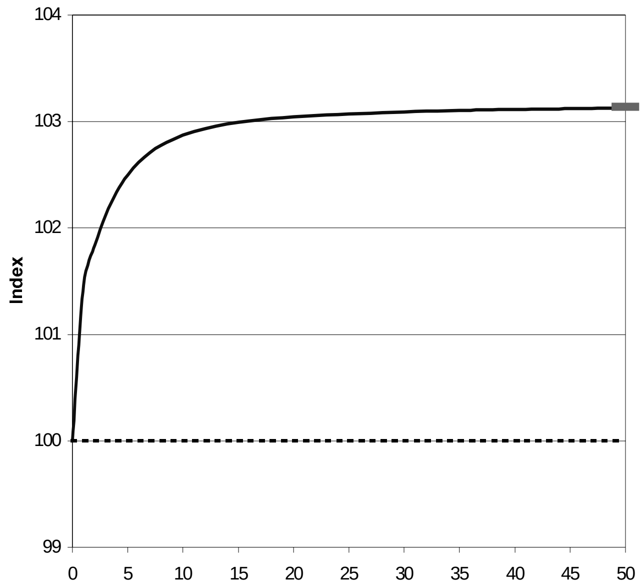

Figure 26 $t^W - 0.05$, Gross output in private sector

E. g. 75 percent of the adjustment in private gross production is reached after 5 periods (25 years), (see figure 26) while the same level of adaptation takes 21 periods (105 years) for the private consumption (see figure 27).

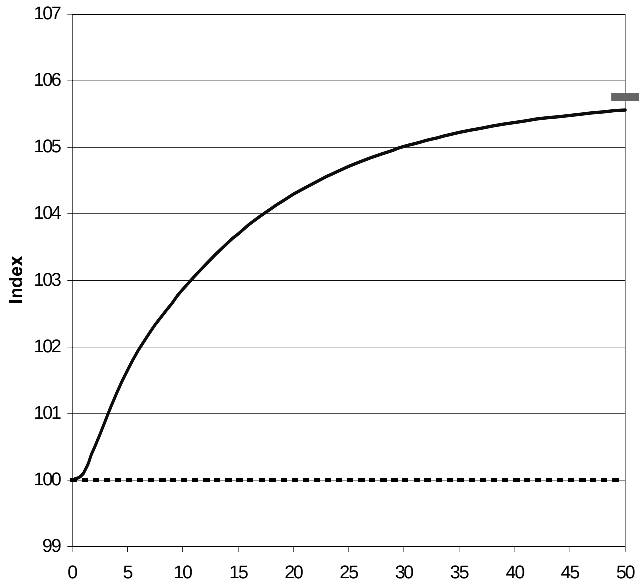

Figure 27 $t^W - 0.05$, Private consumption

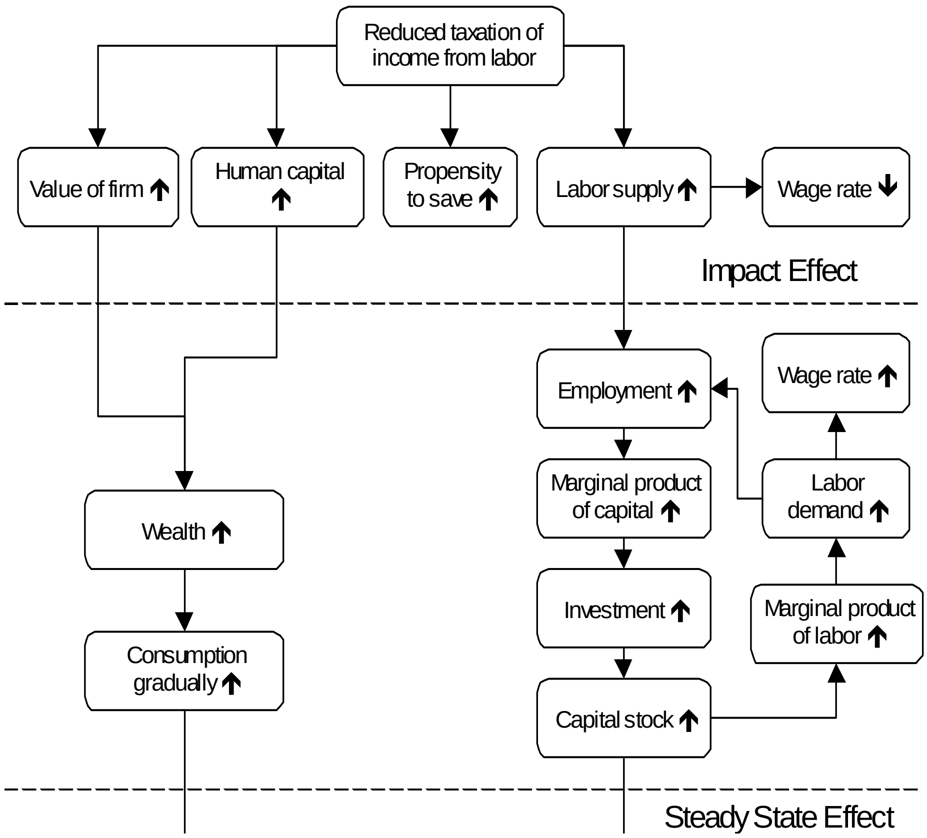

Figure 28 Effects of a reduction in labor taxation

Thus the very slow transition process is first of all due to the long overlaps of generations, rather than to the gradual adjustment process of firms. But also the adjustment of prices through foreign trade and the convex installation costs in the private production are important contributors to prolongation of the transition process. This subject is explored further in sensitivity analyses in section XXX.

The mechanisms that generates both the initial effects and the dynamic transition of the economy are illustrated in figure 28. As already noted there are two main direct effects: A supply side effect through the increase in the labor supply and a demand side effect working through the forward-looking behavior of the agents which implies that non-predetermined stock variables adjust at the moment when the information about the shock is revealed to the agents. The relevant forward-looking stock variables in this case are the value of firms and the stock of human capital.

We describe the effects in detail below.

8.1.1 Supply side effects

In this section we focus upon the right hand side of the flow diagram 28, which represents the supply side effects of the policy experiment. This is divided into the initial effect on the labor supply and the subsequent dynamic effect on the capital stock.

Initial labor supply

A reduction in the marginal wage tax rate increases the marginal wage rate after tax. At the initial wage rate, this implies that each individual will supply more labor, since in optimum the marginal disutility of work must be equal to the marginal benefit from work. Recall that the labor supply is given as

$$\ell_t \equiv \left(\frac{\left[W_t - T_t^w\left(W_t\right)\right] - \left[b_t - T_t^b\left(b_t\right)\right]}{\gamma_1 P_t} \right)^{\gamma}$$

The initial effect on the labor supply can be calculated by inserting the actual numbers in the expression, also using the benefit indexation rule ($b_t = \varphi^b W_t \left(1 - t_t^{\ell}\right)$, see page 60) and fixing the wage rate and the consumer price index at their initial levels. Normalizing the wage rate to 1, the numerator prior to the tax cut amounts to[2]

$$(1 - 0.427) - 0.680 \times (1 - 0.06) \times (1 - 0.306) \simeq 0.1294$$

[2] Confer the calibration chapter for an overview of the tax and indexation parameters.

The tax cut changes the numerator to

$$(1-0.377)-0.680\times(1-0.06)\times(1-0.306)\simeq 0.1794$$

As the denominator is not directly affected by the wage tax cut, the direct initial effect is therefore an increase in the marginal reward of working by $\frac{0.1794-0.1294}{0.1294}\simeq 39$ percent. This directly expands the labor supply by 3.3 percent as the labor supply elasticity, γ, is equal to 0.1. In sum, decreasing the marginal wage tax rate in a standard competitive labor market with an upward sloping labor supply schedule initially shifts the labor supply curve outwards (to the right). This is the effect which initiates the right hand side of the flowchart in figure 28.

Capital accumulation

As mentioned the impact effect of the permanent outward shift in the labor supply schedule is that employment rise and the wage rate is reduced. The resulting increase in employment increases the marginal product of capital and thereby the shadow price of capital deflated by the price index for private investments. This stimulates investments and the capital stock gradually increases. The marginal product of labor is then raised and the demand for labor likewise. This feeds back on the employment and also causes the wage rate to increase again. This explains the dynamic adjustment shown in the right hand side of figure 28. To begin with, we look at the private sector.

To see the relative effects of the different components in the investment decision consider the first order condition with respect to private investments. For convenience the equation is repeated below

$$\frac{1-t_t^d}{1-t_t^g}\left(1-g+(1-t_t^c)\frac{p_t^P}{p_t^{PI}}\frac{\partial\Phi^P}{\partial I^P}(I_t^P,K_{t-1}^P)\right)=\frac{\lambda_{1t}}{p_t^{PI}}+\lambda_{2t}$$

Although the effect here is rather simple we present the figures which decompose the effect on the investment behavior in the following. The left hand side of the first order condition for investments (above) is decomposed in figure 29.

The left hand side of the equation measures the marginal costs of investments. These marginal costs are decomposed into two components that are defined as follows.

$$\text{Direct marginal costs of investments}:\frac{1-t_t^d}{1-t_t^g}(1-g)$$

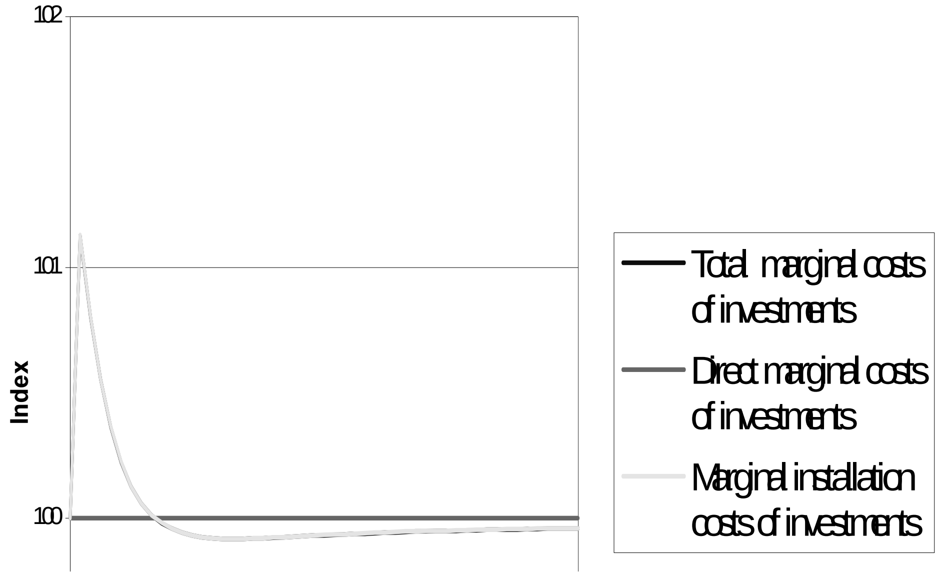

Figure 29 $t^W - 0.05$, Total marginal costs of investments, decomposed

$$\text{Marginal installation costs of investments} : \frac{1-t_t^d}{1-t_t^g}\left((1-t_t^c)\,\frac{p_t^P}{p_t^{PI}}\frac{\partial\Phi^P}{\partial I^P}(I_t^P, K_{t-1}^P)\right)$$

The graph displays the index value of the total costs (100 times experiment/baseline) and for the two items above their contribution to this index, i.e. the value the index would have shown if the other items did not change. One observes that the direct marginal costs of investment does not change in the present experiment and therefore the entire effect on the marginal cost of investments is due to the effect from the marginal installation costs, which implies that the curve representing these costs and the curve representing total costs coincide. Thus the marginal costs of investments increase at first due to increased investments. As capital is accumulated marginal installation costs start to diminish, and investments will rise even further - see also figure 30.

The fact that the private capital stock does not jump to a higher level from the beginning - like the employment[3] is due to the presence of convex installation costs,

[3] Above 90 percent of the adjustment in the employment takes place in the first period following the tax cut, whereas the same level of adjustment for the private capital stock is first reached after 12-13 periods. The private investments takes almost 80 percent of the adjustment in the first period, but this is not even closely sufficient to make the capital stock follow the development in employment, because of the depreciation of physical capital at the rate of about 29 percent every period.

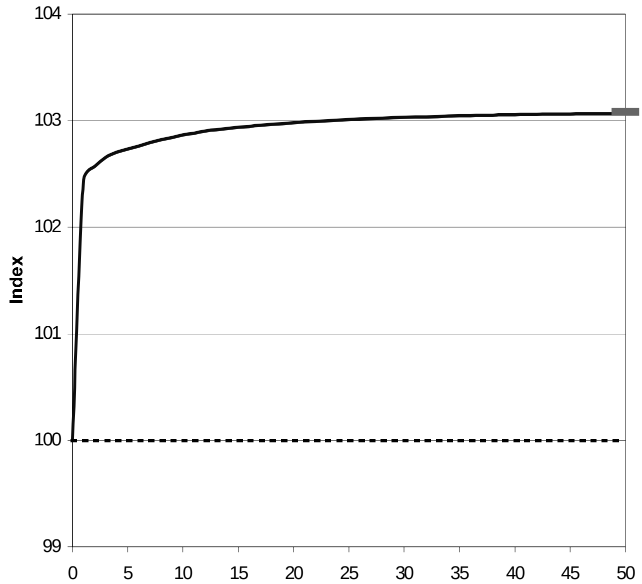

Figure 30 $t^W - 0.05$, Private gross investments

which causes firms to smooth the investments over a longer period. Without the convexity of installation costs (i.e. $\iota = 0$), private investments would initially jump to satisfy the need for capital and then afterwards fall back to a lower level (overshooting) - however still above baseline to sustain the higher level of capital.

Similarly the right hand side of the first order condition, which measures the marginal benefits of investment is decomposed into figure 31 and 32. The benefits consists of the shadow price of capital deflated by the price index for private investments (Tobin's q) and in the shadow price of the book value of capital.

Figure 31 shows that there is an instantaneous increase in Tobin's q, whereas figure 32 shows no effect on the shadow price of the book value of capital. The shadow price of marginal book capital, λ_{2t}, depends only on the rate of interest (net of tax) and tax parameters applied directly to firms, and is therefore unaffected by the wage tax cut - see (2.25) at page 28. The net effect on the marginal gain from investments is therefore equal to the positive effect from the increase in Tobin's q. This positive effect follows from the initial increase in employment which implies an increase in the marginal product of capital. From the first order condition (2.20) the change on impact in the shadow price of capital is seen to be positively correlated to the marginal product of capital. The first order condition of investments repeated above implies that investments are stimulated until the marginal installation costs of investments

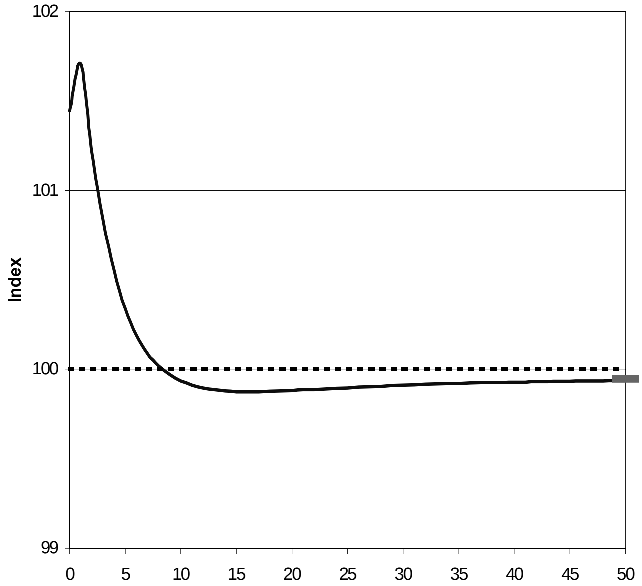

Figure 31 $t^W - 0.05$, Shadow price of marginal capital

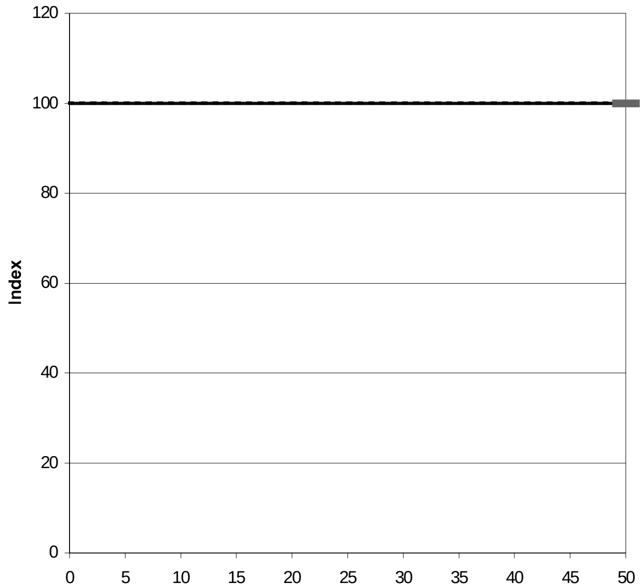

Figure 32 $t^W - 0.05$, Shadow price of marginal book capital are increased accordingly, cf. figure 30, and gradually the capital stock also grows, as shown in figure 33.

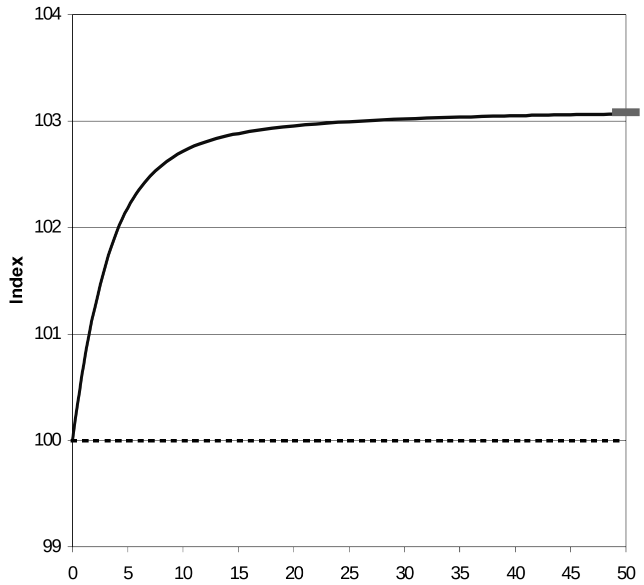

Figure 33 $t^W - 0.05$, Capital stock in private sector

The combination of figure 29 and figure 32 shows that initially both the marginal costs and benefits of investments increase, but in the long run they fall back just below the baseline levels[4], as the capital stock has adapted to the new and higher stationary state level.

In the stationary state the marginal product of capital is only affected by the income tax changes through the price ratio $\frac{p^P}{p^{PI}}$, see relation (6.9). In the case where there was no effect on the price ratio the capital stock would increase only as a result of the effect on the marginal product of capital from the increase in the labor supply. In the present case where the terms of trade is endogenous there is an additional positive effect on the capital stock from the reduced relative price of the domestic production, which again follows from the increased domestic production following the shift in the labor supply curve.

Now turning to the investments in the public sector, the analysis becomes somewhat simpler. This is due to the assumption - made in chapter 2 - that the public sector

[4]The first order condition is satisfied at all times, and therefore the two curves develop in the same way.

does not perform any intertemporal optimization, and instead ties public (gross) investments to a constant share of overall GDP[5].

The initial effect on employment from the wage tax cut is simultaneously transmitted to GDP but since the other input factors adjusts much slower (especially the capital stock), the public sector investments initially only take on a quarter of the total adjustment to the stationary state level. Thus the exogenous nature of the governmental investments prolongs the accommodation in comparison to the private sector adjustment - see figure 34

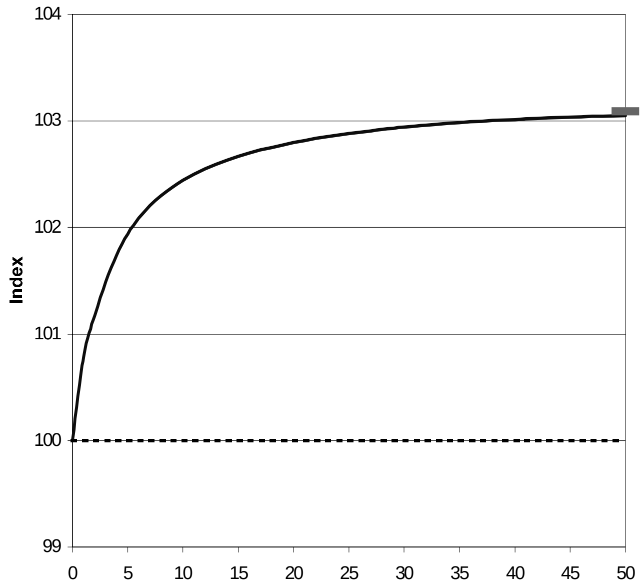

Figure 34 $t^W - 0.05$, Public gross investments

8.1.2 Demand side effects

The demand side effects are initiated by the shift in the value of firms and in the value of human capital in the economy. Before concentrating on these effect we discuss the budgetary effect of the policy, since this reveals the life cycle incentives of the experiment.

[5]Recall that there are no installation costs of investments in the governmental production, so gross and net production is equivalent in this context.

Budgetary effect of the policy

The reduction in the tax rate implies that the ordinary tax revenue, tax_t, to the public sector is reduced - primarily from two sources. First, the effect through LT_t (the tax-revenue from labor income tax on employed) is negative, directly due to the cut on the wage tax rate itself, and indirectly due to the lower equilibrium (pre-tax) wage rate - cf. figure 35.

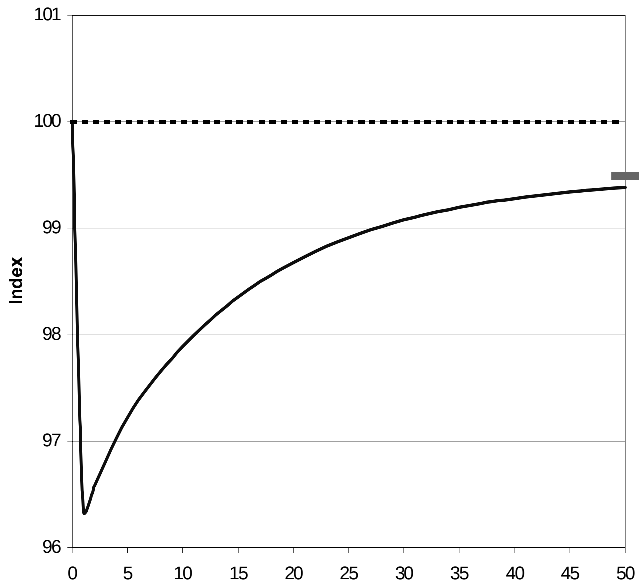

Figure 35 $t^W - 0.05$, Wage rate

However, there is also an indirect and positive effect working through LT_t, which arises from the positive initial effect on employment that ceteris paribus contributes to an increased wage sum and thereby higher taxes. The latter effect is distinctly dominated by the two former, so the total effect is negative. Secondly, the effect through BT_t (the tax-revenue from labor income tax on unemployed) is also negative, since the initial effect on employment reduces the unemployment and since the lower (pre-tax) wage rate causes the benefit rate to fall due to the indexation rule (4.5) - see page 60. Both effects tend to lower the sum of benefits paid out, and finally reducing the collected taxes from this source. The effect through LT_t is about 8 times the size of the effect from BT_t.

Government spending reacts to the reduction in the marginal tax rate on wage income by the mentioned reduction in the spending on unemployment benefits and through an increase in the public consumption which follows from the increase in the activity of the private sector - recall from chapter 5, section 3 that government consumption is indexed to GDP.

The net effect of the changes in government revenue and spending is a deficit that causes the lump sum transfer to the households to decrease so that the budget remains balanced. Recall that the lump sum transfer is identical for all adults. The net initial effect on current disposable income of the generations alive is that generations who belong to the labor force experience a positive effect, whereas the current disposable income of each pensioner declines. The present value of future disposable income flows therefore initially declines for pensioner generations. It increases initially for generations who are young enough for their remaining periods in the labor force to generate additional disposable wage income sufficient to outweigh their discounted future loss as pensioners. This picture of gains and losses initially stimulates the savings of the younger generations to smooth consumption over the life cycle.

To determine the effect on the level of consumption of each generation we need to consider the effect of the policy on the future prices as well as the human and non human wealth of the consumers. Recall that consumption of a generation is given as

$$C_{b,t} = \xi_b^S \left(\frac{1 + r_t \left(1 - t_t^r\right)}{1+\theta} \frac{\eta_{b-1,t-1}}{P_t} N_{b-1,t-1}^{EF} \right)^S \frac{a_{b-1,t-1} + H_{b-1,t-1}}{\eta_{b-1,t-1}} + Z_{b,t}, \quad 18 \leq b \leq 78$$

where $\eta_{b-1,t-1}$ is the index of future consumption prices for generation b and $a_{b-1,t-1}$ is the level of non-human wealth, while $H_{b-1,t-1}$ is the level of human capital. $Z_{b,t}$ is the disutility of work.

Both the human and the non-human capital are forward-looking (jump) variables. Non-human capital is forward-looking because it includes the value of domestic firms, since we assume that there is no foreign ownership of the firms.

The development in the consumer price index is shown in figure 36.

Prices fall by 1.1 percent in the first 5 periods (25 years). This is followed by a gradual increase of 0.9 percent in the rest of the transition period. The dynamic path of prices will tend to give a small incentive to increase consumption in the first periods, but this effect is not very strong. Therefore the crucial effects are those that follow the evolution of the value of firms and the human capital respectively. In the following these effects are decomposed.

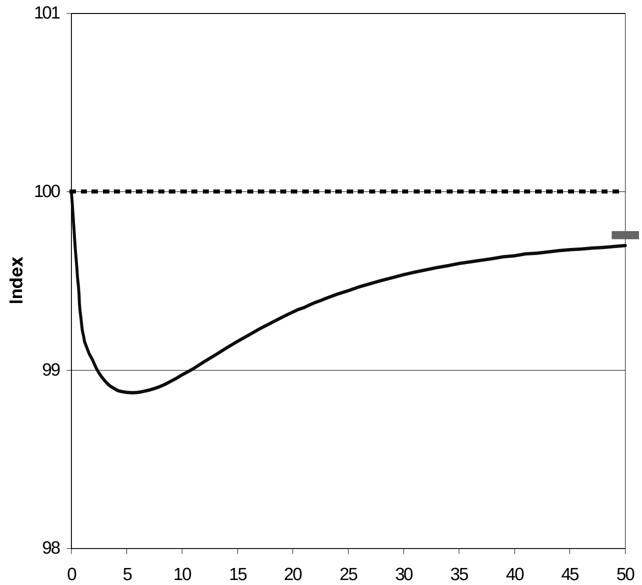

Figure 36 $t^W - 0.05$, Consumer CES price index

Firm value

In this subsection an account of the increase in firm value - displayed in figure 28 - is given. The total dividend, D_t, and the value of firms, V_t, are shown in figure 37 and figure 38, respectively.

The value of the firm is defined as the present discounted value of future dividends of the firms, cf. chapter 2. The diagrams show that the value of firms increases for all periods compared to baseline. Even though the dividends temporarily decline in the first period, the value of firms initially (immediately after the announcement of the tax cut at the end of period zero) jumps to a higher level. This is explained by the fact that dividends in all but the first period is markedly above the baseline level. The positive contributions to the firm value from the discounted value of the dividends of these periods dominate the initial one-period setback.

The repercussion in the firm value around period 5 resembles the profile of the dividends. Note however that the fluctuations are much smaller and arise a few periods in advance. Again this is due to the forward-looking nature of the variable and to the discounting, giving greater weight to dividends in the near future[6].

[6] If the discounting was sufficiently severe, the profile would approximate the profile of dividends.

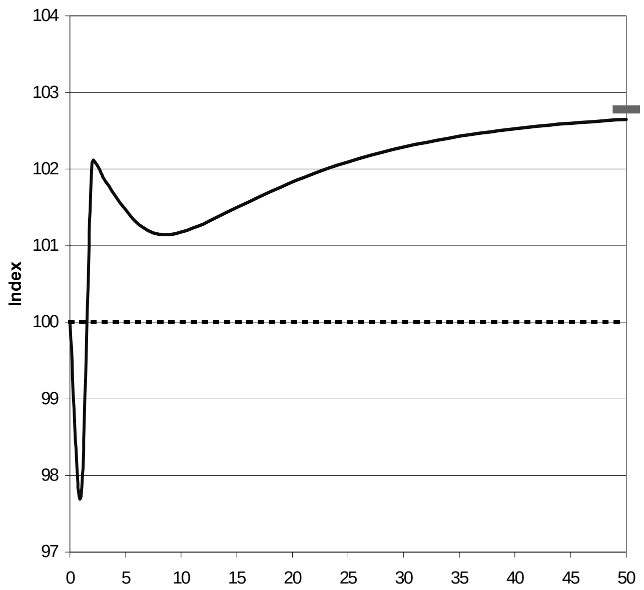

Figure 37 $t^W - 0.05$, Dividend

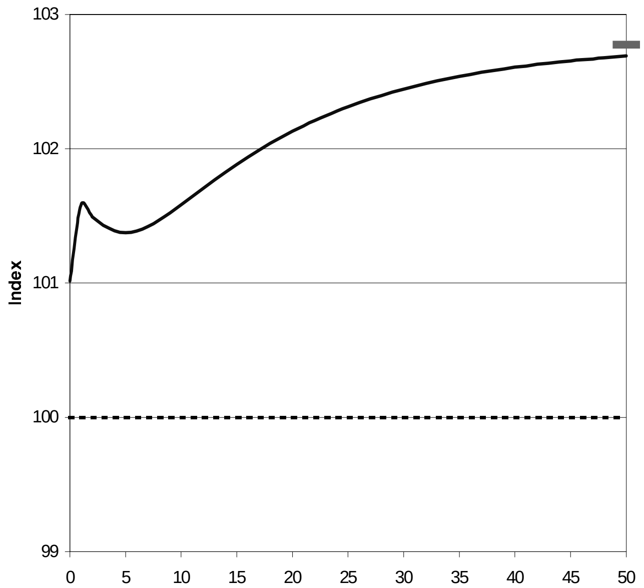

Figure 38 $t^W - 0.05$, Value of firm

To track the effect further we decompose the effect on the stream of dividends.[7]. For convenience the expression for the dividends is repeated below

$$\begin{aligned} D_t &= (1-t_t^c)\left(p_t^P Y_t^P - p_t^{PM} M_t^P - (1+t_t^a) W_t L_t^P - r_t B_{t-1}^c\right) \\ &\quad - p_t^{PI} I_t^P + t_t^c \hat{\delta}_t \hat{K}_{t-1} + \left(B_t^c - B_{t-1}^c\right) \end{aligned}$$

This expression is then split into the following categories

$$\text{Current gross profits: } (1-t_t^c)\left(p_t^P Y_t^P - p_t^{PM} M_t^P - (1+t_t^a) W_t L_t^P\right)$$

$$\text{Interest payments: } -(1-t_t^c)\, r_t B_{t-1}^c$$

$$\text{Investments: } -p_t^{PI} I_t^P$$

$$\text{Tax depreciation allowance: } t_t^c \hat{\delta}_t \hat{K}_{t-1}$$

$$\text{Changed debt: } B_t^c - B_{t-1}^c$$

The evolution over time of the contribution to the stream of dividend of these 5 items are shown in figure 39.

Initially total dividends decline as mentioned before. This is mainly because of the massive expansion of investments, which more than outweigh the increase in the gross production which results from the increase in employment. The gradual increase in the capital stock proportionally increases the corporate debt (due to the fixed debt ratio) and thereby the interest payments, which lowers dividends. However, the increase in corporate debt in itself pulls in the opposite direction. This is also the case for the rise in the current gross profits (brought about by a rise in gross output and diminished labor costs), which is the main contributor to the increase in nominal dividends for the entire period. Gradually the amount of tax depreciation increases. As stationary state is reached the corporate debt becomes stable and there is no effect from change of debt (as by definition also was the case in the baseline).

[7] In the present case this is fairly simple so the following presentation serves primarily as an introduction to the figures that present the decomposition.

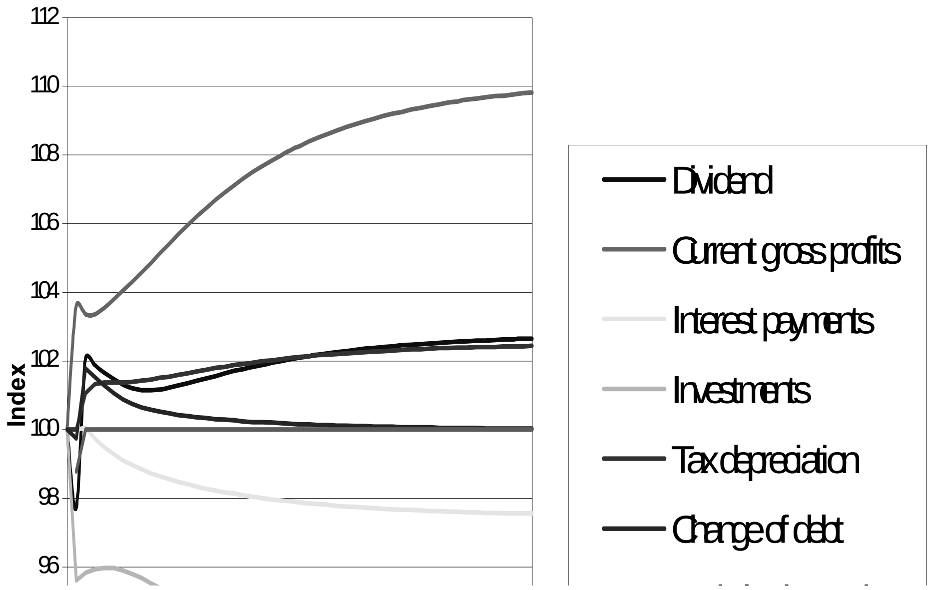

Figure 39 $t^W - 0.05$, Dividend, decomposed

Human capital

The diagram on top of figure 8.1.9 shows the consequences for the present value of disposable income flows for each generation.

The interpretation of the x-axis is somewhat unusual. Recall that new generations of households enter the economy at the end of period, so that generation 0 is the youngest generation (i.e. 18-22 years of age) economically active in period 1, when the policy change is implemented, and generation -11 is the oldest (i.e. 73-77 years of age). The first twelve marks on the x-axis thus show the impact effect on the twelve generations "alive" in period 1, whereas the following points meassure the group of 18-22-year-olds in all future periods. For instance the effect on generation 19 pictured in the diagram is the effect on the generation that is 18-22 years old in period 20. The retirement age is 61 years which in period 1 belongs to the age interval of generation -8 (58-62 years). Roughly 40 percent of generation -8 is assumed to be retired leaving 60 percent in the labor force.

Recall also that human capital is defined as the present value of the current and future disposable income from: wage, unemployment benefits, inheritance, age dependent transfers (both taxable and non-taxable), pensions and lump sum transfers from abroad and from the domestic government - corrected for the disutility value of working.

The diagram on top of figure 8.1.9 shows the index for human capital and the contribution to this from each of these seven items[8]. For the generations living at period 1 (who first appeared at the labor market at period -10 to 1), total human capital declines for pensioners and for generations with few periods left in the labor force (generations from -11 to -6), while the rest of the population experiences an improvement - the younger the higher. For yet unborn generations (i.e. generations > 0), the benefit from the future increases of employment and disposable wage (deflated by the consumer price index) is even bigger.

These generation-specific human capital effects can be given the following interpretation: The fact that older persons experience a lower and even negative effect on their human capital from the increase in the present value of their disposable wage earning is due to the following three facts: First, there is the contractional effect on income from the reduced lump sum transfers from the government. Second, for those currently alive there is an effect from the fact that the older the generation the less the wage earnings contribute to the human capital (cf. figure 21 at page 110), and third there is a macroeconomic effect from the fact that employment grows gradually as the labor demand schedule is shifted outwards with the increased capital stock. The gradual outward shift of the labor demand implies that both the real wage and employment is increased and thus the wage sum of the generations who are active on the labor market is gradually increased. The later (in time) the generation enters the labor market the higher the effect on wage sum for the years in workforce, and therefore the higher is the effect on human capital

It can be seen from figure 8.1.9 that also retired generations (generations -9 and -10) observe (tiny) increases in the present value of their current and future wage earnings. This is due to the wage and benefit income of (the few) men in the labor supplying age (18-60 years) who are married to women of pensioner age (61-77 years) as the age of the women defines the age of the households. For all generations the present value of unemployment benefit earnings decline because unemployment falls - initially by one quarter. Naturally, this contributes negatively to the human capital.

The effect on the pension after tax is negative since pensioners - by the definition of the experiment - do not experience a tax-cut. Pensions before tax are regulated

[8] Since lump sum transfers from abroad are exogenous this item is simply put together with the domestic lump sum transfers for reason of clarity.

according to the developments in the pre-tax wage. Since this wage initially falls as a consequence of the wage tax cut, so does the pension income. From figure 8.1.9 one observes that the effect from pensions on human capital is larger the older the generation. This is composed of two effects. First, the older the generation the larger the share of pension in total human capital of the given generation. Second, the younger the generation the smaller is the discount factor attached to the reduction in pensions after tax and the higher is the wage - and thereby the pension - at the time when they reach retirement age.

Age dependent transfers - both taxable and non-taxable - are fixed to the wage rate before tax, which initially falls by close to 4 percent (see figure 35) and then gradually recovers. Therefore their present value will be most negatively affected by the oldest generation, who only has one period left as economically active. When future generations will be born the age dependent transfers has almost returned to their baseline level (0.5 percent below) and the present value of these will therefore only be short of the baseline level by a little amount. The described fluctuations are however not very visible in the decomposition of real human capital, since the contribution from age dependent transfers is so relatively small.

Turning to inheritance one should first look at the

Adding up these generational human capital effects to find the effect on aggregate present value of disposable income (not shown in the decomposition) would reveal an initial reduction followed by a gradual increase over time - to a level markedly above baseline.

The aggregate consumption and utility

In the preceding 2 subsections the contribution to the wealth of the agents from the forward-looking stock variables was accounted for. As the consumption demand is determined by the wealth and the evolution in prices over time we may now turn to a discussion of the generation specific consumption and the aggregate consumption.

First, the increase in the value of firms implies an unforeseen capital gain of those generations alive at the implementation of the policy. The distribution of gains across generations follows the distribution of the stock of non-human capital across generations. Thus it is implicitly assumed that the composition of wealth is identical across generations. Second, the effect on the present value of disposable income is negative for elder generations alive and positive for the youngest generations alive. These facts (along with the very small effect from price changes) lead to the conclusion that on impact demand for consumption is reduced for the oldest half of the population and increased for the youngest half. Aggregate consumption is practically unchanged on impact.

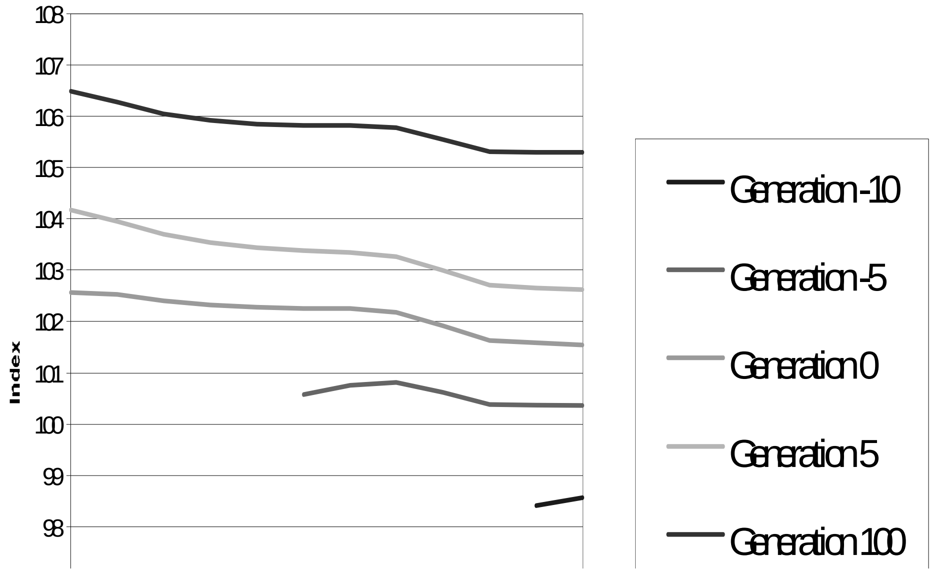

Figure 40 Consumption for generations

The evolution of the counterfactual generation-specific life cycle consumption path relative to the baseline life cycle consumption path is shown in figure 40, where it is shown that generation -10 experience a reduction in consumption in the remaining lifetime after the policy experiment. The opposite is true for generation -5 and later generations[9]. The later the generation enters the economy the higher the increase in the life cycle consumption path. Aggregate (private) consumption is shown in figure 27.

Finally consider the welfare effect of the different generations. The pensioners do not work therefore the welfare effect of these generations is similar to the effects on their life cycle consumption path. Therefore their welfare deteriorate. The same is true for the oldest generation in the labor force, since their consumption is also reduced

[9] Generation -6 (not shown in the diagram) experiences a higher consumption in the first 3 periods following the tax cut, but then a tiny reduction sets in for the last 3 periods of life (i.e. age 63-77) in comparison to baseline. Consumption levels for generations from -11 to -7 all deteriorate for the rest of their economically active lives.

and in addition to this they have increased labor supply which increases disutility of work. The latter effect is small compared to the consumption effect in the current calibration of the model. Therefore younger generations experience an increase in utility along with their increased life cycle consumption. The diagram at the bottom of figure 8.1.9 shows the welfare changes across generations.

Chapter 9
APPENDIX

9.1 Appendix A - notation of population variables

This appendix is meant to give an understanding of the notational structure of the population variables. The form can be summarized into the following five rules of thumb:

1. All population variables are denoted by a capital letter N.

2. The last subscript is always referring to the time (except in stationary state where time can be ignored).

3. Superscript letters is referring to the gender

F	$\sim$	females
M	$\sim$	males

or to the type

A	$\sim$	adults
C	$\sim$	children
E	$\sim$	(adult)equivalents
H	$\sim$	heirs
W	$\sim$	workers

4. When the first (gender)superscript is followed by a W we are referring to the part of the gender who is working, and when it is followed by an H to the part who is heirs at the given time. Both the first and second superscript can be followed by an F. In that case we are referring to the part of the men or the type or both in combination assigned to a specific household/age of the female.

5. When there is an arrow above the variable ($\vec{N}$) then it is a vector otherwise just a number.

The definitions of all the population variables are listed explicitly below

N_t	$\sim$	total population
$N_{i,t}$	$\sim$	population of age i
		$N_t = \sum_{i=0}^{\inf} N_{i,t}$
N_t^M	$\sim$	total number of men at time t
$N_{a,t}^M$	$\sim$	number of men aged a at time t
$N_{b,t}^{MF}$	$\sim$	number of men assigned to household b at time t
$N_{a,b,t}^{MF}$	$\sim$	number of men of age a assigned to household b at time t
		$N_t^M = \sum_{a=18}^{\tilde{A}} N_{a,t}^M$
		$N_{a,b,t}^{MF} = N_{a,t}^M \cdot \omega_{a,b}^M$
		$N_{b,t}^{MF} = \sum_{a=18}^{\tilde{A}} N_{a,b,t}^{MF}$
N_t^F	$\sim$	total number of females at time t
$N_{b,t}^F$	$\sim$	number of females aged b at time t
		$N_t^F = \sum_{b=18}^{78} N_{b,t}^F$
N_t^C	$\sim$	total number of children converted into adult-equivalents measured at time t
$N_{c,t}^C$	$\sim$	number of children aged c converted into adult-equivalents measured at time t
$N_{b,t}^{CF}$	$\sim$	number of children (converted into adult-equivalents) assigned to household b at time t
$N_{c,b,t}^{CF}$	$\sim$	number of children (converted into adult-equivalents) aged c assigned to household b at time t
		$N_t^C = \sum_{c=0}^{17} N_{c,t}^C$
		$N_{c,b,t}^{CF} = N_{c,t}^C \cdot \psi_{c,b}$
		$N_{b,t}^{CF} = \sum_{c=0}^{17} N_{c,b,t}^{CF}$
N_t^A	$\sim$	total number of adults at time t
$N_{b,t}^{AF}$	$\sim$	number of adults assigned to household b at time t
		$N_t^A = N_t^M + N_t^F$
		$N_{b,t}^{AF} = N_{b,t}^{MF} + N_{b,t}^F$
N_t^E	$\sim$	total number of adult-equivalents at time t
$N_{b,t}^{EF}$	$\sim$	number of adult-equivalents assigned to household b at time t
		$N_t^E = N_t^A + N_t^C$
		$N_{b,t}^{EF} = N_{b,t}^{AF} + N_{b,t}^{CF}$

N_t^W ~ total number of workers at time t
N_t^{MW} ~ total number of working men at time t
$N_{a,t}^{MW}$ ~ number of working men aged a at time t
$N_{b,t}^{MWF}$ ~ number of working men assigned to household b at time t
$N_{b,t}^{FW}$ ~ number of working females aged b at time t
$N_{b,t}^{WF}$ ~ number of workers assigned to household b at time t

$$N_{a,t}^{MW} = \begin{cases} N_{a,t}^{M} & for\ 18 \leq a < 61 \\ 0 & for\ a \geq 61 \end{cases}$$

$$N_{b,t}^{MWF} = \sum_{a=18}^{60} N_{a,b,t}^{MF}$$

$$N_{b,t}^{FW} = \begin{cases} N_{b,t}^{F} & for\ 18 \leq b < 61 \\ 0 & for\ b \geq 61 \end{cases}$$

$$N_{b,t}^{WF} = N_{b,t}^{FW} + N_{b,t}^{MWF}$$

$$= \begin{cases} N_{b,t}^{F} + \sum_{a=18}^{60} N_{a,b,t}^{MF} & for\ 18 \leq b < 61 \\ \sum_{a=18}^{60} N_{a,b,t}^{MF} & for\ a \geq 61 \end{cases}$$

N_t^H ~ total number of heirs at time t
$N_{h,t}^H$ ~ number of heirs aged h at time t
$N_{a,t}^{MH}$ ~ number of male heirs aged a at time t
$N_{b,t}^{MHF}$ ~ number of male heirs assigned to household b at time t
$N_{a,b,t}^{MHF}$ ~ number of male heirs aged a assigned to household b at time t
$N_{b,t}^{FH}$ ~ number of female heirs aged b at time t

$$N_t^H = \sum_{h=29}^{60} N_{h,t}^H$$

$$N_{h,t}^H = N_{h,t}^{MH} + N_{h,t}^{FH}$$

$$N_{a,b,t}^{MHF} = N_{a,t}^{MH} \cdot \omega_{a,b}^{M}$$

$$N_{b,t}^{MHF} = \sum_{a=29}^{60} N_{a,b,t}^{MHF}$$

N_t^O ~ total number of very old persons at time t

$$N_t^O = \sum_{b=78}^{\tilde{B}} N_{b,t}^F + \sum_{a=78}^{\tilde{A}} N_{a,t}^M - \sum_{b=18}^{77} \sum_{a=78}^{\tilde{A}} N_{a,t}^M \cdot \omega_{a,b}$$

$\vec{N}_t^{M}$	$\sim$	row vector $\left(1 \times \left(\tilde{A} - 18\right)\right)$ consisting of the elements $N_{a,t}^{M}$, $a = 18, 19, ..., \tilde{A}$ at time t
$\vec{N}_t^{MF}$	$\sim$	row vector (1×60) consisting of the elements $N_{b,t}^{MF}$, $b = 18, 19, ..., 77$ at time t
$\vec{N}_t^{F}$	$\sim$	row vector (1×60) consisting of the elements $N_{b,t}^{F}$, $b = 18, 19, ..., 77$ at time t
$\vec{N}_t^{C}$	$\sim$	row vector (1×18) consisting of the elements $N_{c,t}^{C}$, $c = 0, 1, ..., 17$ at time t
$\vec{N}_t^{CF}$	$\sim$	row vector (1×60) consisting of the elements $N_{b,t}^{CF}$, $b = 18, 19, ..., 77$ at time t
$\vec{N}_t^{AF}$	$\sim$	row vector (1×60) consisting of the elements $N_{b,t}^{AF}$, $b = 18, 19, ..., 77$ at time t
$\vec{N}_t^{EF}$	$\sim$	row vector (1×60) consisting of the elements $N_{b,t}^{EF}$, $b = 18, 19, ..., 77$ at time t
$\vec{N}_t^{29-60}$	$\sim$	row vector (1×32) consisting of the elements $N_{i,t}$, $i = 29, 30, ..., 60$ at time t
		$\vec{N}_t^{AF} = \vec{N}_t^{MF} + \vec{N}_t^{F}$
		$\vec{N}_t^{EF} = \vec{N}_t^{AF} + \vec{N}_t^{CF}$

9.2 Appendix B - CES functions

In this appendix we will analyze the CES-function

$$F(x_1, ..., x_n) = \left[\sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}} \right]^{\frac{E}{E-1}}, \ \beta_j > 0, \ j = 1, ..., n \tag{9.1}$$

In what follows, we will make extensive use of the price index

$$P \equiv \frac{\sum_{j=1}^{n} p_j x_j}{F(x_1, ..., x_n)} \tag{9.2}$$

9.2.1 Cost minimization

We are considering the following problem

$$\begin{array}{rl} (x_1, ..., x_n) Min \ C \equiv \sum_{j=1}^{n} p_j x_j \\ s.t. \quad \left[\sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}} \right]^{\frac{E}{E-1}} = \bar{V} \end{array} \tag{9.3}$$

In our context this problem has two interpretations: if the agent is a producer, $\bar{V}$ is the level of production, and if the agent is a consumer, then $\bar{V}$ is the level of utility.

Theorem 1 *The solution to problem (9.3) is*

$$x_j = \beta_j^E \left(\frac{p_j}{P} \right)^{-E} \bar{V}, \ where \tag{9.4}$$

$$P = \left[\sum_{j=1}^{n} \beta_j^E p_j^{1-E} \right]^{\frac{1}{1-E}} \tag{9.5}$$

Proof. The following Lagrange-function represents the problem

$$L \equiv \sum_{j=1}^{n} p_j x_j - \lambda \left(\sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}} - \bar{V}^{\frac{E-1}{E}} \right)$$

Differentiation yields

$$\frac{\partial L}{\partial x_j} = p_j - \lambda \frac{E-1}{E} \beta_j x_j^{\frac{E-1}{E}-1} = 0$$

such that

$$p_j x_j = \lambda \frac{E-1}{E} \beta_j x_j^{\frac{E-1}{E}} \tag{9.6}$$

Summation implies

$$\sum_{j=1}^{n} p_j x_j = \lambda \frac{E-1}{E} \sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}} = \lambda \frac{E-1}{E} \bar{V}^{\frac{E-1}{E}}$$

Solving for λ, substituting this into (9.6) and rearranging implies

$$x_j = \left(\frac{\beta_j}{p_j} \frac{\sum_{i=1}^{n} p_i x_i}{\bar{V}} \right)^E \bar{V} \tag{9.7}$$

For $F(x_1, ..., x_n) = \bar{V}$, substitution of the index defined by (9.2) gives $(*)$.

To find the specific form of the price index, rearrange $(*)$

$$\beta_j x_j^{\frac{E-1}{E}} = \beta_j^E \left(\frac{p_j}{P} \right)^{1-E} \bar{V}^{\frac{E-1}{E}}$$

such that

$$\sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}} = \bar{V}^{\frac{E-1}{E}} \sum_{j=1}^{n} \beta_j^E \left(\frac{p_j}{P} \right)^{1-E}$$

Using the restriction in (9.3) we have

$$1 = \sum_{j=1}^{n} \beta_j^E \left(\frac{p_j}{P} \right)^{1-E}$$

which yields $(**)$. $\square$

9.2.2 Utility maximization

We are considering the following problem

$$\begin{array}{l} (x_1, ..., x_n) Max\ U = \left[\sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}}\right]^{\frac{E}{E-1}} \\ s.t. \quad \sum_{j=1}^{n} p_j x_j = \bar{R} \end{array} \tag{9.8}$$

Theorem 2 *The solution to the problem (9.8) is*

$$x_j = \beta_j^E \left(\frac{p_j}{P}\right)^{-E} \frac{\bar{R}}{P}, \text{ where} \tag{9.9}$$

$$P = \left[\sum_{j=1}^{n} \beta_j^E p_j^{1-E}\right]^{\frac{1}{1-E}} \tag{9.10}$$

Proof. Obviously, utility maximization implies cost minimization given the optimal utility level. Let U^* be this optimal utility level. Then, according to the theorem on cost minimization

$$x_j = \beta_j^E \left(\frac{p_j}{P}\right)^{-E} U^*, \text{ where} \tag{9.11}$$

$$P = \left[\sum_{j=1}^{n} \beta_j^E p_j^{1-E}\right]^{\frac{1}{1-E}} \tag{9.12}$$

Multiplying with p_j on both sides of (9.11), summarizing and using (9.12), implies

$$\bar{R} = \sum_{j=1}^{n} p_j x_j = PU^* \sum_{j=1}^{n} \beta_j^E \left(\frac{p_j}{P}\right)^{1-E} = PU^*$$

which proves (▲) and (▲▲). □

9.2.3 Profit maximization

Let p be the output price, Y the production, $(p_1, ..., p_n)$ the input prices and $(x_1, ..., x_n)$ the input quantities. We are considering the following problem

$$\begin{array}{l} (x_1, ..., x_n) Max\ \Pi \equiv pY - \sum_{j=1}^{n} p_j x_j \\ s.t. \quad Y = \left[\sum_{j=1}^{n} \beta_j x_j^{\frac{E-1}{E}}\right]^{\frac{E}{E-1}} \end{array} \tag{9.13}$$

Theorem 3 *There is only a well defined solution to the problem (9.13) if*

$$p = P = \left[\sum_{j=1}^{n} \beta_j^E p_j^{1-E}\right]^{\frac{1}{1-E}} \text{. In this case} \tag{9.14}$$

$$x_j = \beta_j^E \left(\frac{p_j}{P}\right)^{-E} Y \text{ , where } Y \text{ is any positive number} \tag{9.15}$$

Proof. Obviously profit maximization implies cost minimization for a given production level. This gives (♦♦) directly, according to the theorem on cost minimization. From (9.2) we have that $\sum_{j=1}^{n} p_j x_j = PY$, such that

$$\Pi = pY - PY = (p - P)Y$$

If $p \neq P$ then $Y = +\infty$ or $Y = -\infty$ is optimal. Therefore a necessary condition for a well defined solution is (♦). □

9.2.4 Calibration

Define a *consistent data set* by $d = (P^0, Y^0, p_1^0, ..., p_n^0, x_1^0, ..., x_n^0)$ where

$$P^0 Y^0 = \sum_{j=1}^{n} p_j^0 x_j^0 \tag{9.16}$$

A vector of coefficints $(\beta_1, ..., \beta_n)$ is said to *calibrate* a CES-function for given elasticity of substitution E and consistent data set d if :

$$x_j^0 = \beta_j^E \left(\frac{p_j^0}{P^0}\right)^{-E} Y^0, \text{ where} \tag{9.17}$$

$$P^0 = \left[\sum_{j=1}^{n} \beta_j^E \left(p_j^0\right)^{1-E}\right]^{\frac{1}{1-E}} \tag{9.18}$$

Theorem:

$(\beta_1, ..., \beta_n)$ is calibrating a CES-function for the elasticity of substitution E and the consistent data set $d = (P^0, Y^0, p_1^0, ..., p_n^0, x_1^0, ..., x_n^0)$ if:

$$\beta_j = \frac{p_j^0}{P^0}\left(\frac{x_j^0}{Y^0}\right)^{\frac{1}{E}} \tag{9.19}$$

Proof: Rearranging (9.19) implies (9.17). Substituting (9.19) into (9.18) implies:

$$\begin{aligned} P^0 &= \left[\sum_{j=1}^{n}\left(\frac{p_j^0}{P^0}\right)^E \frac{x_j^0}{Y^0}\left(p_j^0\right)^{1-E}\right]^{\frac{1}{1-E}} \\ &= \left(P^0\right)^{-\frac{E}{1-E}}\left(Y^0\right)^{-\frac{1}{1-E}}\left[\sum_{j=1}^{n} p_j^0 x_j^0\right]^{\frac{1}{1-E}} \end{aligned}$$

As the data set d is consistent we have that:

$$P^0 = \left(P^0\right)^{-\frac{E}{1-E}}\left(Y^0\right)^{-\frac{1}{1-E}}\left(P^0 Y^0\right)^{\frac{1}{1-E}} = P^0$$

□

Remark 1: The calibrations in DREAM are done by combining (9.16) with (9.19):

$$\beta_j = \frac{p_j^0}{P^0}\left(\frac{P^0 x_j^0}{\sum_{i=1}^{n} p_i^0 x_i^0}\right)^{\frac{1}{E}}$$

Data is often give as a set of national account revenues $(R_1, ..., R_n)$. In this situation, prices (including the price index P^0) can be choosen freely. If $(p_1^0, ..., p_n^0, P^0)$ has been choosen, the $x_j^{0\prime}$s are calculated by :

$$x_j^0 \equiv \frac{R_j}{p_j^0},$$

such that:

$$\beta_j = \left(\frac{p_j^0}{P^0}\right)^{\frac{E-1}{E}}\left(\frac{R_j}{\sum_{i=1}^{n} R_i}\right)^{\frac{1}{E}}$$

It is often assumed that $p_1^0 = ... = p_n^0 = P^0 = 1$. In this case:

$$\beta_j = \left(\frac{R_j}{\sum_{i=1}^{n} R_i}\right)^{\frac{1}{E}}$$

9.3 Appendix C - calculations and derivations

Chapter 6

Signs of the derivatives *Equation* (6.20)

From the first order conditions (6.1), (6.2) and (6.8) we have

$$F'_L + p\left(F''_{LM}\frac{dm}{dp} + F''_{LK}\frac{dk}{dp}\right) = (1+t^a)\frac{dW}{dp} \tag{9.20}$$

$$F'_M + p\left(F''_{MM}\frac{dm}{dp} + F''_{MK}\frac{dk}{dp}\right) = \frac{dp^M}{dp} \tag{9.21}$$

$$F'_K + p\left(F''_{KM}\frac{dm}{dp} + F''_{KK}\frac{dk}{dp}\right) = \frac{dp^I}{dp}\beta^{ss} \tag{9.22}$$

The price-indexes are in the stationary state defined as

$$p^M = \left(1+t^M\right)\left((1-\mu_M)^{\sigma_M} + (\mu_M)^{\sigma_M}(p)^{1-\sigma_M}\right)^{\frac{1}{1-\sigma_M}} \tag{9.23}$$

$$p^I = \left(1+t^I_t\right)\left((1-\mu_I)^{\sigma_I} + (\mu_I)^{\sigma_I}(p)^{1-\sigma_I}\right)^{\frac{1}{1-\sigma_I}} \tag{9.24}$$

Rewriting (9.22) yields

$$\frac{dm}{dp} = \frac{1}{F''_{KM}}\left(\frac{dp^I}{dp}\frac{\beta^{ss}}{p} - \frac{F'_K}{p} - F''_{KK}\frac{dk}{dp}\right)$$

Inserting this exression into (9.21) and reducing, yields

$$\frac{dk}{dp} = \frac{\left(\frac{dp^M}{dp} - F'_M - \frac{F''_{MM}}{F''_{KM}}\left(\frac{dp^I}{dp}\beta^{ss} - F'_K\right)\right)F''_{KM}}{p\left(\left(F''_{MK}\right)^2 - F''_{KK}F''_{MM}\right)} \tag{9.25}$$

To determine the sign of $\frac{dk}{dp}$ in (9.25) consider first the numerator: Insering the first order conditions (6.8) and (6.2) yields

$$\left(\frac{dp^M}{dp} - \frac{p^M}{p} - \frac{F''_{MM}}{F''_{KM}}\left(\frac{dp^I}{dp} - \frac{p^I}{p}\right)\beta^{ss}\right)F''_{KM}$$

From the definition (9.23) it is by simple computation possible to establish that

$$\frac{p^M}{p} > \frac{dp^M}{dp}$$

and similarly for p^I.

By the assumptions concerning the derivatives of 2. order to the production function: $F''_{xx} < 0$, $F''_{xy} > 0$ for $x \neq y$ and $x, y \in \{K, L, M\}$, we have that the numerator is negative.

The sign of the denominator of the expression for $\frac{dk}{dp}$ in (9.25) is determined by the following agument[1]: The gross-production function is homogenous of degree one:

$$F(K, L, M) = KF'_K + LF'_L + MF'_M$$

This implies that

$$F'_K = F'_K + KF'_{KK} + LF''_{LK} + MF''_{MK}$$

$$F'_M = KF'_{KM} + LF''_{LM} + F'_M + MF''_{MK}$$

such that

$$0 = KF''_{KK} + LF''_{LK} + MF''_{MK}$$

$$0 = KF''_{KM} + LF''_{LM} + MF''_{MM}$$

therefore

$$\begin{aligned} F''_{KK}F''_{MM} &= \left(\frac{-LF''_{LK} - MF''_{MK}}{K}\right)\left(\frac{-KF''_{MK} - LF''_{LM}}{M}\right) = \\ & \frac{1}{M}LF''_{LK}F''_{MK} + \frac{1}{KM}L^2F''_{LK}F''_{LM} + (F''_{MK})^2 + \frac{1}{K}F''_{MK}LF''_{LM} \end{aligned}$$

The denominator, $p\left((F''_{MK})^2 - F''_{KK}F''_{MM}\right)$ of $\frac{dK}{dp}$ in expression (9.25) is therefore given as

$$p\left((F''_{MK})^2 - F''_{KK}F''_{MM}\right) = -p\left(\frac{1}{M}LF''_{LK}F''_{MK} + \frac{1}{KM}L^2F''_{LK}F''_{LM} + \frac{1}{K}F''_{MK}LF''_{LM}\right)$$

By assumption the cross marginalproducts are positive, which implies that the denominator is negative, such that

[1]We thank Michael Andersen for providing this argument.

$$\frac{dk}{dp} > 0$$

Condition (9.21) implies

$$\frac{dm}{dp} = \frac{1}{F''_{MM}} \left(\frac{1}{p} \left(\frac{dp^M}{dp} - F'_M \right) - F''_{MK} \frac{dk}{dp} \right)$$

Inserting the first order condtion (6.2) yields

$$\frac{dm}{dp} = \frac{1}{F''_{MM}} \left(\frac{1}{p} \left(\frac{dp^M}{dp} - \frac{p^M}{p} \right) - F''_{MK} \frac{dk}{dp} \right) \Rightarrow$$

$$\frac{dm}{dp} > 0$$

Condition (9.20) implies

$$\frac{dW}{dp} - \frac{1}{(1+t^a)} F'_L = \frac{1}{(1+t^a)} p \left(F''_{LM} \frac{dm}{dp} + F''_{LK} \frac{dk}{dp} \right)$$

Inserting the first order condition (9.22) yields

$$\frac{dW}{dp} - \frac{W}{p} = \frac{1}{(1+t^a)} p \left(F''_{LM} \frac{dm}{dp} + F''_{LK} \frac{dk}{dp} \right) \Rightarrow$$

$$\frac{dW}{dp} > 0$$

Since the real wage response to a price change is defined as

$$\frac{d}{dp} \left(\frac{W}{p} \right) = \frac{\frac{dW}{dp} p - W}{p^2} = \frac{1}{p} \left(\frac{dW}{dp} - \frac{W}{p} \right)$$

it follows that

$$\frac{d}{dp} \left(\frac{W}{p} \right) > 0$$

Individual consumption net of disutility from work *Equations* (6.27)−(6.29)

From the Keynes-Ramsey rule (4.37) we have

$$\frac{Q_{b+1}}{Q_b} = \left(\frac{1+r\left(1-t^r\right)}{1+\theta}\right)^S \Rightarrow Q_b = Q_{18}\left(\frac{1+r\left(1-t^r\right)}{1+\theta}\right)^{S(b-18)} \ , \ 18 < b < 78 \tag{9.26}$$

Using this together with (??) leads to an expression for the stationary state bequest

$$Q_{78} = \xi^S \left(\frac{1+r\left(1-t^r\right)}{1+\theta}\right)^S Q_{77} = Q_{18}\xi^S \left(\frac{1+r\left(1-t^r\right)}{1+\theta}\right)^{S\cdot 60} \tag{9.27}$$

Now defining the aggregate consumption net of disutility from work, Q, and inserting (9.26) into this expression yields

$$\begin{aligned} Q &\equiv \sum_{b=18}^{77} N_b^{EF} Q_b = N_{18}^{EF} Q_{18} + \sum_{b=19}^{77} N_b^{EF} Q_{18} \left(\frac{1+r\left(1-t^r\right)}{1+\theta}\right)^{S(b-18)} \\ &= Q_{18}\left(N_{18}^{EF} + \sum_{b=19}^{77} N_b^{EF}\left(\frac{1+r\left(1-t^r\right)}{1+\theta}\right)^{S(b-18)}\right) \qquad \Leftrightarrow \\ Q_{18} &= \alpha_{18} Q \end{aligned} \tag{9.28}$$

where

$$\alpha_{18} \equiv \frac{1}{N_{18}^{EF} + \sum_{b=19}^{77} N_b^{EF}\left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S(b-18)}}$$

Inserting (9.28) into (9.26) yields

$$Q_b = \alpha_b Q \ , \ \ 18 < b < 78 \tag{9.29}$$

where

$$\alpha_b \equiv \frac{\left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S(b-18)}}{N_{18}^{EF} + \sum_{b=19}^{77} N_b^{EF}\left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S(b-18)}} \ , \ \ 18 < b < 78$$

Inserting (9.28) into (9.27) yields

$$Q_{78} = \alpha_{78} Q$$

where

$$\alpha_{78} \equiv \xi^S \frac{\left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S \cdot 60}}{N_{18}^{EF} + \sum_{b=19}^{77} N_b^{EF} \left(\frac{1+r(1-t^r)}{1+\theta}\right)^{S(b-18)}}$$

Income per adult *Equation* (6.42)

The income per adult in the stationary state in the household of age b can from (4.3) be written as

$$
\begin{aligned}
y_b &= \frac{1}{N_b^{AF}} \left\{ N_b^{FW} \left(\left[W - T^w(W) \right] \ell + \left[b - T^b(b) \right] \left(\bar{\ell} - \ell \right) + B_b^{inF} \right) \right. \\
&\quad + \left(N_b^F - N_b^{FW} \right) \left[f^P - T^P\left(f^P\right) \right] + N_b^F TR_b^F + \left[W - T^w(W) \right] \sum_{a=18}^{60} N_a^M \cdot \omega_{a,b} \cdot \ell \\
&\quad + \left[b - T^b(b) \right] \sum_{a=18}^{60} N_a^M \cdot \omega_{a,b} \cdot \left(\bar{\ell} - \ell \right) + \sum_{a=18}^{\tilde{A}} N_a^M \cdot \omega_{a,b} \cdot TR_a^M \\
&\quad \left. + \sum_{a=29}^{60} N_a^M \cdot \omega_{a,b} \cdot B_a^{inM} + \left[f^P - T^P\left(f^P\right) \right] \sum_{a=61}^{\tilde{A}} N_a^M \cdot \omega_{a,b} \right\} + \tau^W + \tau \\
&= \frac{1}{N_b^{AF}} \left\{ N_b^{FW} \left(\left[W - T^w(W) - b + T^b(b) \right] \ell + \left[b - T^b(b) \right] \bar{\ell} + B_b^{inF} \right) \right. \\
&\quad + \left(N_b^F - N_b^{FW} \right) \left[f^P - T^P\left(f^P\right) \right] + N_b^F TR_b^F \\
&\quad + \left[W - T^w(W) - b + T^b(b) \right] \ell \sum_{a=18}^{60} N_a^M \cdot \omega_{a,b} \\
&\quad + \left[b - T^b(b) \right] \bar{\ell} \sum_{a=18}^{60} N_a^M \cdot \omega_{a,b} + \sum_{a=18}^{\tilde{A}} N_a^M \cdot \omega_{a,b} \cdot TR_a^M \\
&\quad \left. + \sum_{a=29}^{60} N_a^M \cdot \omega_{a,b} \cdot B_a^{inM} + \left[f^P - T^P\left(f^P\right) \right] \sum_{a=61}^{\tilde{A}} N_a^M \cdot \omega_{a,b} \right\} + \tau^W + \tau \\
&= \frac{N_b^{WF}}{N_b^{AF}} \left(\left[W - T^w(W) - b + T^b(b) \right] \ell + \left[b - T^b(b) \right] \bar{\ell} \right) + \frac{N_b^{AF} - N_b^{WF}}{N_b^{AF}} \left[f^P - T^P\left(f^P\right) \right] \\
&\quad + \frac{1}{N_b^{AF}} \left(N_b^{FW} b_b^{inF} + \sum_{a=29}^{60} N_a^M \cdot \omega_{a,b} \cdot b_a^{inM} \right) \alpha_{78} N_{77}^{EF} PQ \quad + \bar{y}_b + \tau + \tau^W
\end{aligned}
$$

where the part $\bar{y}_b$ is exogenous

$$
\bar{y}_b \equiv \frac{N_b^F}{N_b^{AF}} TR_b^F + \frac{1}{N_b^{AF}} \sum_{a=18}^{\tilde{A}} N_a^M \cdot \omega_{a,b} \cdot TR_a^M
$$

and

$$
b_b^{inF} \equiv \frac{B_b^{inF}}{\alpha_{78} N_{77}^{EF} PQ} \,, \quad b_a^{inM} \equiv \frac{B_a^{inM}}{\alpha_{78} N_{77}^{EF} PQ}
$$

Aggregate human capital *Equation* (6.46)

The income adjusted for the disutility of work, $\tilde{y}_b$ given by (4.17), can then be calculated from the income terms above

$$\begin{aligned}
\tilde{y}_b &= y_b \frac{N_b^{AF}}{N_b^{EF}} - PZ_b = y_b \frac{N_b^{AF}}{N_b^{EF}} - P\frac{N_b^{WF}}{N_b^{EF}} f(\ell) \\
&= \frac{N_b^{WF}}{N_b^{EF}} \left\{ \left[W - T^w(W) - b + T^b(b)\right]\ell + \left[b - T^b(b)\right]\overline{\ell} - Pf(\ell) \right\} \\
&\quad + \frac{N_b^{AF} - N_b^{WF}}{N_b^{EF}} \left[f^P - T^P\left(f^P\right)\right] + \frac{1}{N_b^{EF}} \left(N_b^{FW} b_b^{inF} + \sum_{a=29}^{60} N_a^M \cdot \omega_{a,b} \cdot b_a^{inM} \right) \alpha_{78} N_{77}^{EF} PQ \\
&\quad + \frac{N_b^{AF}}{N_b^{EF}} \left(\overline{y}_b + \tau + \tau^W\right)
\end{aligned}$$

We are now able to deduce an expression for the aggregate human capital

$$\begin{aligned}
H &= \sum_{b=18}^{77} N_b^{EF} H_b = \sum_{b=18}^{77} N_b^{EF} \sum_{i=b+1}^{77} \tilde{y}_i \tilde{R}_{b,i} \\
&= \sum_{b=18}^{77} N_b^{EF} \Bigg\{ \left\{ \left[W - T^w(W) - b + T^b(b)\right]\ell + \left[b - T^b(b)\right]\overline{\ell} - Pf(\ell) \right\} \sum_{i=b+1}^{77} \frac{N_i^{WF}}{N_i^{EF}} \tilde{R}_{b,i} \\
&\quad + \left[f^P - T^P\left(f^P\right)\right] \sum_{i=b+1}^{77} \frac{N_i^{AF} - N_i^{WF}}{N_i^{EF}} \tilde{R}_{b,i} \\
&\quad + \alpha_{78} N_{77}^{EF} PQ \sum_{i=b+1}^{77} \left(\frac{N_i^{FW}}{N_i^{EF}} b_i^{inF} + \frac{1}{N_i^{EF}} \sum_{a=29}^{60} N_a^M \; \omega_{a,i} \; b_a^{inM} \right) \tilde{R}_{b,i} \\
&\quad + \sum_{i=b+1}^{77} \frac{N_i^{AF}}{N_i^{EF}} \left(\overline{y}_i + \tau + \tau^W\right) \tilde{R}_{b,i} \Bigg\} \\
&= \left\{ \left[W - T^w(W) - b + T^b(b)\right]\ell + \left[b - T^b(b)\right]\overline{\ell} - Pf(\ell) \right\} N^W \sum_{b=18}^{77} \frac{N_b^{EF}}{N^W} \sum_{i=b+1}^{77} \frac{N_i^W}{N_i^{EF}} \tilde{R}_{b,i} \\
&\quad + \left[f^P - T^P\left(f^P\right)\right] \left(N^A - N^W\right) \sum_{b=18}^{77} \frac{N_b^{EF}}{N^A - N^W} \sum_{i=b+1}^{77} \frac{N_i^{AF} - N_i^{WF}}{N_i^{EF}} \tilde{R}_{b,i} \\
&\quad + \alpha_{78} N_{77}^{EF} PQ \sum_{b=18}^{77} N_b^{EF} \sum_{i=b+1}^{77} \left(\frac{N_i^{FW}}{N_i^{EF}} b_i^{inF} + \frac{1}{N_i^{EF}} \sum_{a=29}^{60} N_a^M \cdot \omega_{a,i} \cdot b_a^{inM} \right) \tilde{R}_{b,i} \\
&\quad + N^A \sum_{b=18}^{77} \frac{N_b^{EF}}{N^A} \sum_{i=b+1}^{77} \frac{N_i^{AF}}{N_i^{EF}} \overline{y}_i \tilde{R}_{b,i} + \left(\tau + \tau^W\right) N^A \sum_{b=18}^{77} \frac{N_b^{EF}}{N^A} \sum_{i=b+1}^{77} \frac{N_i^{AF}}{N_i^{EF}} \tilde{R}_{b,i} \\
&= f_1 \left\{ \left[W - T^w(W) - b + T^b(b)\right]\ell + \left[b - T^b(b)\right]\overline{\ell} - Pf(\ell) \right\} N^W \\
&\quad + f_2 \left[f^P - T^P\left(f^P\right)\right] \left(N^A - N^W\right) + f_3 PQ + \left(f_4 \left(\tau + \tau^W\right) + \bar{h}\right) N^A
\end{aligned}$$

where the help-variables f_1, f_2, f_3, f_4 and $\bar{h}$ used in the last step are defined below

$$
\begin{aligned}
f_1 &\equiv \sum_{b=18}^{77} \frac{N_b^{EF}}{N^W} \sum_{i=b+1}^{77} \frac{N_i^{WF}}{N_i^{EF}} \tilde{R}_{b,i} \ , \\
f_2 &\equiv \sum_{b=18}^{77} \frac{N_b^{EF}}{N^A - N^W} \sum_{i=b+1}^{77} \frac{N_i^{AF} - N_i^{WF}}{N_i^{EF}} \tilde{R}_{b,i} \ , \\
f_3 &\equiv \alpha_{78} N_{77}^{EF} \sum_{b=18}^{77} N_b^{EF} \sum_{i=b+1}^{77} \left(\frac{N_i^{FW}}{N_i^{EF}} b_i^{inF} + \frac{1}{N_i^{EF}} \sum_{a=29}^{60} N_a^M \cdot \omega_{a,i} \cdot b_a^{inM} \right) \tilde{R}_{b,i} \ , \\
f_4 &\equiv \sum_{b=18}^{77} \frac{N_b^{EF}}{N^A} \sum_{i=b+1}^{77} \frac{N_i^{AF}}{N_i^{EF}} \tilde{R}_{b,i} \ , \\
\bar{h} &\equiv \sum_{b=18}^{77} \frac{N_b^{EF}}{N^A} \sum_{i=b+1}^{77} \left(\frac{N_i^F}{N_i^{EF}} TR_i^F + \frac{1}{N_i^{EF}} \sum_{a=18}^{\tilde{A}} N_a^M \cdot \omega_{a,i} \cdot TR_a^M \right) \tilde{R}_{b,i}
\end{aligned}
$$

9.4 Appendix D - population prediction

This appendix describes how the population is forecasted and constructed. This process can be divided into 2 steps:

1: Forecast the population for n number of years

2: Construct households, i.e. assign men and children to women

9.4.1 Forecasting the population

Forecasting the population is a recursive process that depends on the initial population as well as assumptions about in- and outflow from the population. First recall that $N_{a,t}^M$ and $N_{a,t}^F$ are the number of men and women (respectively) aged a years at time t. It will be useful to create the set S that consist of the two genders, thus S=$\{M, F\}$ (do not confuse the set S with the elasticity of substitution used in the consumer's utility function).

Mortality

We need to define age and gender specific mortality rates. Let $\left(1 - \Theta_{a,t,S}^{\dagger}\right)$ be the survival frequency for individuals aged a[2] years and of gender S at time t. Thus,

[2]Note that a is an element of the set A, that is the set of ages admissible, i.e. $a \in A = \{0, 1, 2, ...99\}$. It is thus assumed that persons do not life more than 100 years. The set $\hat{A} \subset A$, is a proper subset containing the ages when women are fertile.

$\left(1 - \Theta^{\dagger}_{a,t,S}\right)$ is the fraction of a year old (of gender S) that live to experience their $a+1$ years birthday at time t+1. If this number is one for a given age a then all individuals survive to turn $a+1$ years. If the number is zero then everyone dies. It is assumed nobody experience their 100 years birthday, and thus $\Theta^{\dagger}_{99,t,S} = 1$ for S=$\{M, F\}$.

Fertility

We also need to define age-specific fertility rates for women in the fertile ages. Let $\Theta^{B}_{a,t}$ be defined at the number of children that an a years old woman gives birth to at time t. Note that this number is assumed to be independent of the number of men in the population. We will also define an age independent fertility, which we will call F_t. The final parameter we need to define is the fraction of newborns that are of the two genders. We will call this fraction $\varsigma_{t,S}$; it follows that $\varsigma_{t,M}+\varsigma_{t,F} = 1$ at all times.

Migration

Finally we need to define net-migration at time t for individuals of age a and gender S as $NM_{a,t,S}$. This number may be positive or negative, and is considered exogenous.

The population forecast model

To get started forecasting population, we need initial values, i.e. the population in 1995. The equations below describe the population forecast model. The first equation takes hand of mortality and migration:

$$N_{a+1,t+1,S} = \left(1 - \Theta^{\dagger}_{a,t,S}\right) \cdot N_{a,t,S} + NM_{a,t,S} \quad (\forall\, a \in A\backslash\{99\} \,\wedge\, t \geqq 1995) \tag{9.30}$$

The second equation takes hand of fertility, and determines the number of newborn by sex:

$$N_{0,t,S} = \varsigma_{t,S}\left(F_t + \sum_{a \in \hat{A}} N_{a,t}\Theta^{B}_{a,t}\right) \qquad (\forall t \geqq 1996 \,\wedge\, S = \{M, F\}) \tag{9.31}$$

Together these two equations describe the complete system of motion for the population, and for given initial values ($N_{a,1995,S}$) it is easy to calculate the population at any given time recursively.

The population forecast used in the model is particularly simple, since it is assumed that

- the mortality is constant through time, i.e. $\Theta^{\dagger}_{a,t,S}$ reduces to $\Theta^{\dagger}_{a,S}$
- there is no migration, i.e. $NM_{a,t,S} = 0$

With these simplifications equation (9.30) reduces to

$$N_{a+1,t+1,S} = \left(1 - \Theta^{\dagger}_{a,S}\right) \cdot N_{a,t,S} \, (\forall \, a \in A \backslash \{99\} \, \wedge \, t \geqq 1995)$$

Furthermore we assume that

- fertility is not age dependent: $\Theta^{B}_{a,t} = 0$
- fertility is constant and time independent (i.e. $F_{t,S} = F_S$).
- The ratio between boys and girls among the newborns is constant, i.e. $\varsigma_{t,S} = \varsigma_S$.

Thus (9.31) reduces to

$$N_{0,t,S} = \varsigma_S \cdot F_S \qquad (\forall t \geqq 1996)$$

9.4.2 Constructing households

The method outlined above forecasts the population - the next step is to construct the households that constitute the economic decision making unit. The idea behind this is that each woman is assigned to a distribution of men and a distribution of children. This procedure was explained in section @.@.

In chapter @.@ it was explained how men and women were matched using the couple matrix Ω_{1995}. However, there is a problem with truncation in the data; the number of couples where either the man or the woman are below 18 are reported as zero. Thus the distribution is truncated from below[3]. For women aged e.g. 55 this problem is not large, since their number of male partners below 18 years is small - but for women aged 18 the problem is obvious.

The solution adopted is to assume that the *relative* partner age-pattern for women aged 18, is the same as for the woman aged 28. In other words we assume, that the fraction of m year old men joined with 28 years old women, are the same as the fraction of m-10 year old men that are joined with 28-10=18 years old women. This generalization is made for women between 18 and 27 years. In practice this alleviates the truncation problem[4].

Let us now turn towards the men. Consider the group of men that are 18 years old. Clearly some of the women with whom they are associated, are below 18 years. In this case we have, that even though *they* (the men) are over 18 years and thus adults, the woman (=household) to which they belong is below 18 years - and thus still living at home.

The same holds for older men. Some of the women with whom they are associated, are above 77 years, and have retired - this means that they belong to a retired household and thus themselves become retired. These two groups should be subtracted from the number of men "available" to women between 18 and 77 years $\left(N_{a,t}^{M}\right)$.

The number of men in this situation (i.e. where their partner is either a minor or is retired) is illustrated as the difference between the two curves in the figure below

The two effects mentioned above are present in the figure, and cause the dotted line to lie below the solid line. First, there is the number of men joined with women that are minor; this accounts for the effect in the left part of the figure. Secondly, there are a number of men associated with a retired woman; this accounts for the effect in the right-hand side of the figure. The numbers behind this figure can be seen from table 1 below. It shows for each generation the total number of men and the number of men that are available to women between 18 and 77 years - the remaining have

[3] In principle the distribution is also truncated from above, since couples where the man is older than 99 years are ignored. This problem, however, is manageable since we only are interested in women below 78 years. It is not ureasonable to assume, that the number of couples where the woman is 77 years and the man older than 100 is very small.

[4] In principle, of cause, the truncation problem is still present - we have just pushed it 10 years. However, the magnitude of the problem is reduced, since there only is a small number of couples where the age difference is larger than 10 years (less than 1 per cent of 50 years old women are "married" with men below 40 years).

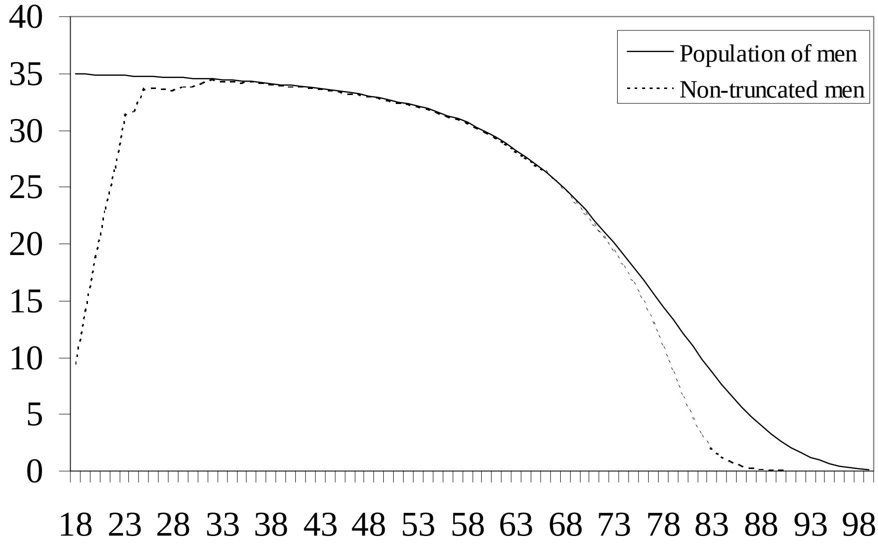

Figure 37 Male population in steady state (thousands)

been subtracted.

	Population	"Available"	Available in percent
18	34.973	9.549	27%
19	34.947	14.211	41%
20	34.916	18.861	54%
21	34.884	23.206	67%
22	34.856	26.788	77%
23	34.828	31.390	90%
24	34.796	31.649	91%
25	34.765	33.513	96%
26	34.732	33.806	97%
27	34.697	33.702	97%
28	34.662	33.600	97%
29	34.625	33.843	98%
30	34.587	33.913	98%
31	34.546	34.194	99%
32	34.500	34.500	100%
33	34.447	34.371	100%
34	34.391	34.315	100%
35	34.332	34.254	100%
36	34.269	34.269	100%
37	34.200	34.200	100%
38	34.125	34.125	100%
39	34.042	33.967	100%
40	33.949	33.873	100%
41	33.845	33.845	100%
42	33.739	33.739	100%
43	33.642	33.642	100%
44	33.538	33.538	100%
45	33.426	33.426	100%
46	33.306	33.237	100%
47	33.179	33.179	100%
48	33.039	33.039	100%
49	32.875	32.874	100%
50	32.690	32.690	100%
51	32.501	32.501	100%
52	32.303	32.302	100%
53	32.093	32.093	100%
54	31.859	31.858	100%
55	31.592	31.591	100%
56	31.306	31.306	100%
57	31.004	30.999	100%

	Population	"Available"	Available in percent
58	30.661	30.657	100%
59	30.262	30.254	100%
60	29.833	29.825	100%
61	29.357	29.351	100%
62	28.821	28.809	100%
63	28.250	28.228	100%
64	27.657	27.618	100%
65	27.033	26.986	100%
66	26.348	26.284	100%
67	25.584	25.512	100%
68	24.779	24.564	99%
69	23.933	23.648	99%
70	23.036	22.657	98%
71	22.119	21.632	98%
72	21.145	20.509	97%
73	20.114	19.293	96%
74	19.036	17.962	94%
75	17.933	16.517	92%
76	16.802	14.911	89%
77	15.639	13.036	83%
78	14.481	10.866	75%
79	13.316	8.583	64%
80	12.164	6.397	53%
81	11.032	4.541	41%
82	9.873	3.084	31%
83	8.746	2.024	23%
84	7.668	1.304	17%
85	6.653	820	12%
86	5.692	514	9%
87	4.809	319	7%
88	4.013	200	5%
89	3.298	123	4%
90	2.652	75	3%
91	2.094	45	2%
92	1.639	27	2%
93	1.251	16	1%
94	932	10	1%
95	662	6	1%
96	452	3	1%
97+	648	3	0%
Total	**1.951.418**	**1.773.166**	**91%**

Table 1: Men in population and men "available" to women between 18 and 77 years

What implication does this have for the analysis presented in this paper? It is important to realize, that whenever the number of men $N_{a,t}^M$ are referred to, we refer to the dotted line since it reflects the number of men "available" to women between 18 and 77 years. An implication of this, is that some men "disappear" in the sense, that they are either matched with a minor or with a retired woman. One might ask why $N_{a,t}^M$ is constructed in this way, instead of just adding two columns to the couple matrix reflecting the truncated area (i.e. a category with women below 18 years, and a category with women over 78 years). The answer is, that for a number of purposes in the computer program and in the calculations, it is convenient that each row in the normalized couple-matrix $\left(\bar{\Omega}_t\right)$ sums to unity for women between 18 and 77 years.

More formally define the extended couple matrix $\hat{\Omega}_t$ by

$$\hat{\Omega}_t = \begin{bmatrix} \hat{N}^{couple}_{0,0,t} & \cdots & \hat{N}^{couple}_{0,18,t} & \hat{N}^{couple}_{0,19,t} & \cdots & \hat{N}^{couple}_{0,77,t} & \cdots & \hat{N}^{couple}_{0,\tilde{B},t} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{N}^{couple}_{18,0,t} & \cdots & \hat{N}^{couple}_{18,18,t} & \hat{N}^{couple}_{18,19,t} & \cdots & \hat{N}^{couple}_{18,77,t} & \cdots & \hat{N}^{couple}_{18,\tilde{B},t} \\ \hat{N}^{couple}_{19,0,t} & \cdots & \hat{N}^{couple}_{19,18,t} & \hat{N}^{couple}_{19,19,t} & \cdots & \hat{N}^{couple}_{19,77,t} & \cdots & \hat{N}^{couple}_{19,\tilde{B},t} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{N}^{couple}_{\tilde{A},0,t} & \cdots & \hat{N}^{couple}_{\tilde{A},18,t} & \hat{N}^{couple}_{\tilde{A},19,t} & \cdots & \hat{N}^{couple}_{\tilde{A},77,t} & \cdots & \hat{N}^{couple}_{\tilde{A},\tilde{B},t} \end{bmatrix}, \quad (9.32)$$

Comparing this matrix to 3.1 it is seen that the matrix is merely extended - the original couple matrix Ω_t is contained in the box (note the that the entries are now called $\hat{N}$). The new matrix takes account for all possible age-combinations, some of which clearly are absurd.

To formalize how the number of non-available men are calculated, consider one row in the matrix. Now define the ratio $\hat{N}^{odd}_{a,t}$ as

$$\hat{N}^{odd}_{a,t} = \frac{\sum_{b=0}^{17} \hat{N}^{couple}_{a,b,t} + \sum_{b=78}^{\tilde{B}} \hat{N}^{couple}_{a,b,t}}{\sum_{b=0}^{\tilde{B}} \hat{N}^{couple}_{a,b,t}}$$

In other words $\hat{N}^{odd}_{a,t}$, is the number of couples where the man is a years old and the age of the woman lies outside the interval $[18, 77]$. This is the fraction of men we label as "unavailable" (to the 18-77 years old women). The value of $1 - \hat{N}^{odd}_{a,Steady-State}$, i.e. the available men in steady state, was shown in table 1 above. Thus the number of men we distribute to women by matrix multiplication is $\left(1 - \hat{N}^{odd}_{a,t}\right) \hat{N}^{M}_{a,t}$ and not $N^{M}_{a,t}$.

Constructing the households in the model using the principles mentioned above[5], the final composition of the households are shown in table 2 below.

[5] Children are distributed according to the age of their mother, as described in section @.@.

	Women	Men	Children
18	33.359	35.555	181
19	33.350	36.133	591
20	33.336	35.566	1.196
21	33.323	35.188	1.989
22	33.314	35.051	3.054
H1	**166.683**	**177.493**	**7.011**
23	33.305	34.471	4.475
24	33.294	34.009	6.223
25	33.284	33.356	8.320
26	33.273	32.917	10.738
27	33.258	32.788	13.553
H2	**166.413**	**167.541**	**43.308**
28	33.245	32.944	16.663
29	33.231	33.434	19.772
30	33.215	33.927	22.559
31	33.199	34.432	25.074
32	33.181	34.786	27.182
H3	**166.070**	**169.522**	**111.251**
33	33.161	35.025	28.911
34	33.141	35.047	30.378
35	33.117	35.017	31.505
36	33.089	34.906	32.164
37	33.055	34.711	32.439
H4	**165.562**	**174.706**	**155.398**
38	33.016	34.468	32.297
39	32.977	34.265	31.841
40	32.934	33.972	30.986
41	32.881	33.628	29.678
42	32.823	33.210	27.990
H5	**164.630**	**169.543**	**152.792**
43	32.763	32.742	25.912
44	32.697	32.236	23.493
45	32.629	31.797	20.674
46	32.549	31.531	17.559
47	32.459	31.528	14.443
H6	**163.097**	**159.833**	**102.081**

	Women	Men	Children
48	32.361	31.875	11.653
49	32.254	32.476	9.141
50	32.138	32.983	7.040
51	32.008	33.441	5.321
52	31.868	33.586	3.872
H7	**160.629**	**164.362**	**37.026**
53	31.721	33.480	2.762
54	31.557	33.207	1.935
55	31.379	32.699	1.260
56	31.187	32.129	801
57	30.966	31.276	467
H8	**156.811**	**162.789**	**7.224**
58	30.728	30.480	257
59	30.478	29.425	133
60	30.195	28.599	59
61	29.877	27.510	27
62	29.540	26.256	13
H9	**150.818**	**142.269**	**489**
63	29.183	25.699	6
64	28.795	24.579	1
65	28.369	23.943	
66	27.903	23.388	
67	27.410	22.469	
H10	**141.661**	**120.079**	**7**
68	26.884	21.762	
69	26.325	20.725	
70	25.745	19.711	
71	25.142	18.511	
72	24.475	17.346	
H11	**128.571**	**98.057**	**-**
73	23.751	16.006	
74	23.014	14.692	
75	22.245	13.432	
76	21.412	12.085	
77	20.516	10.757	
H12	**110.939**	**66.973**	**-**

Table 2: Composition of households in steady state

Having "removed" a large number of younger men (i.e. most prominently men younger than 23 years), it is not immediately obvious why the number of men in the young households seem to be unaffected. But remember that men on average are some years older than their female counterpart - and the men we removed are associated with women younger than 18.

The couplematrix over time

...

Distributing men over time

[HER MANGLER NOGET OM PARMATRICEN OVER TID]

9.5 Appendix E - data

This appendix documents the data for the year of 1995. The main source is the national account figures from the old version, but various other sources are utilized as well. In October 1997, Statistics Denmark published a new national account based on the ENS95 standard. This will of course be utilized in the future versions of DREAM. At present the new version of the national account is not yet complete and therefore the old version is utilized instead.

9.5.1 Input output data

Table 1 shows the basic input output table for 1995. The latest published input output table of the old version from Statistics Denmark covers 1992. However, all figures of the non-shaded areas of table 1 are obtained from the preliminary old version national account figures for 1995 in the database of ADAM (Statistic Denmark's macroeconometric model, cf. Dam (1996)). The figures of the shaded areas are imputed by the so-called RAS-procedure. First, initial guesses of the shaded cells are calculated by applying the corresponding columnwise coefficients of the 1992 input output table to the known (non-shaded) marginals of the 1995 table. Then the shaded cells of the 1995 table are found by minimizing the sum of squared relative deviations to these initial guesses under the restrictions that the marginal figures are not affected.[6]

Finally, re-exports amounting to 32.067 billion Dkr. are consolidated out to avoid the need for modelling re-exports in the model. The omission of re-exports has no impacts on the aggregate activity as it diminishes both imports and exports by the same amount leaving net exports and GDP unaffected. The amounts of imports include toll. The total toll revenue was 1.936 bill. Dkr. in 1995.[7]

[6] ADAM's database contains a preliminary input output table for 1995 in 1980-prices. However, it does not contain a balanced input output table in current prices, which we need here.

[7] Strictly speaking, "commodity taxes" of the column for domestic production include some minor duties not linked to commodity inputs.

	Domestic production		Consumption		Gross investments		Exports	Sum
	Private	Public	Private	Public	Private	Public		
1.a. Domestic private production	437793	59986	312254	0	106769	14039	311187	1242028
1.b. Domestic public production	7653	5798	18928	243905	0	0	216	276500
2.a. Imports of private production	143293	8804	85205	0	28756	1451	0	267509
2.b. Imports of public production	0	0	0	0	0	0	0	0
3.a. Quantity commodity taxes	0	0	29500	0	0	0	0	29500
3.b. Ad valorem commodity taxes, net	7526	18081	73235	0	9630	4011	-4433	108050
4. Uses at market prices (1-3)	596265	92669	519122	243905	145155	19501	306970	1923587
5. Compensation of employees	341383	173776						
6. Gross operating surplus	304380	10055						
7. Gross output at basic values (4-6)	1242028	276500						

Note: Reexports consolidated out.
Source: *ADAM's databank*.

Table 1. The base year input output table, mill. Dkr.

9.5.2 Capital stock and depreciation rates

The national account provides two alternative measures of the capital stock: The gross capital stock and the net capital stock. The gross capital stock is defined as the value of all capital goods evaluated at replacement prices (prices on new investment goods). The net capital stock is the value of all capital goods evaluated at depreciated replacement prices, thereby taking their remaining lifetimes into account. If two capital goods posses the same quality but have different remaining lifetimes, they will have the same value in gross terms, but in net terms the good with the largest remaining lifetime will have the largest value.

The gross concept only corrects for the physical decay of past investments. It is a measure of production capacity and therefore relevant in production functions. The net concept further corrects for the economic decay and is therefore a measure of wealth. It can be shown that in case of a constant physical rate of depreciation, the two measures are identical because in this instance the physical and economic decay coincide. DREAM assumes a constant physical rate of depreciation and there is only one concept of capital in the model. However this does not exempt one for choosing one of the capital concepts for calibration. For the entire capital stock, the gross measure exceeds the net measure by more than 70 per cent. Obviously, the national account is far from assuming a constant physical rate of depreciation.

We prefer the net capital stock for calibration purposes for two reasons. First, in an OLG model measuring the wealth of households accurately must have top priority. Second, the consequences of measuring the capital stock in the production function inadequately are minor. Ideally, the production function should be specified with the services of the capital stock as input. In lack of data for these services, the capital stock is instead inserted as an input, assuming that the services are proportional to the stock. At a given point in time, measuring the stock inaccurately amounts to redefining the factor of proportionality. Of course, this line of argument does not alleviate the basic problem that the model for pure convenience simply assumes a constant physical decay, while the national account based on careful examination of reality assumes a rather different pattern. This implies that even if the capital stock is properly measured in the base year data set, the physical depreciations and therefore the capital stock will develop inaccurately over time in model simulations.

Table 2 display some capital stock figures. The "5 year value" is the ordinary "1 year value" divided by 5, which is utilized when the model is specified with a 5 year frequency to keep the relation between stocks and flows unchanged, when we adopt the convention of keeping the size of flows equal in 1 year and 5 year frequencies as explained in chapter 7.

	1 year value	5 year value
Private sector exclusive dwellings, net	1113.822	223
Dwellings, net	1393.044	279
Total private sector, net	2506.866	501
Public sector, net	522.33	104
Memo		
Total capital stock, net	3029.196	606
Total capital stock, gross	5271.443	1054

Sources: *ADAM's database*.

Table 2. The value of the capital stock, bill. Dkr.

The national account estimates the physical depreciations in 1995 to 133.676 billion Dkr. for the private producers and 10.055 mill. Dkr. for the governmental producers. The base year rates of depreciations for the two sectors are therefore

$$
\begin{aligned}
\delta^P_{base} &= \frac{133.676}{2506.866} = 0.0533 \\
\delta^G_{base} &= \frac{10.055}{522.330} = 0.0193
\end{aligned}
$$

9.5.3 Tax rate of depreciation

The tax rate of depreciation is 0.3 per cent for machinery and transport equipment. The tax depreciation formula for buildings and construction is not a constant rate, but for the calibration of the Danish OLG-model EPRU it is estimated that the tax rules can be translated into an estimated constant tax rate of depreciation for buildings at 0.112.[8] The share of machinery and transport equipment of the total private capital stock (including dwellings) is 0.1696, giving the weighted average estimate of the aggregate tax rate of depreciation for the private producers

$$\widehat{\delta}_{t_0} = 0.1696 \cdot 0.30 + (1 - 0.1696) \cdot 0.112 = 0.1439$$

9.5.4 Public finances

Table 3 gives an overview of the public accounts comparing the benchmark data set and the base year data set. The public (or governmental) sector in DREAM is defined as the state and local governments including The Social Pension Fund and The Central Bank. In 1995 there was a public budget deficit of 23.911 bill. Dkr. as kan be seen from the base year data. In spite of this it is necessary to introduce lump sum transfers at 30.403 bill. Dkr. from the public sector to the households in order to balance the public budget in the benchmark data set. There are 5 major reasons for this as table 4 reveals

[8] The calculation is documented in an unpublished internal paper by Schultz Møller (1993).

		Benchmark data set		Base year data set
Revenue				
Gross operating surplus		10.055		10.055
Wage tax	219.973			
Benefits tax	11.821			
Age dependent transfers tax	14.007			
Pensions tax	23.432			
Payroll tax	0.000			
Interest tax	43.106			
Dividends tax	19.353			
Corporate tax	28.566			
Capital gains tax	0.000			
Total direct taxes etc.		360.260		322.241
Quantity tax private consumption	29.500		29.500	
Ad valorem tax private consumption	73.235		73.235	
Materials excise tax	25.607		25.607	
Investments excise tax	13.641		13.641	
Exports excise tax	-4.433		-4.433	
Tariff	1.936		1.936	
Total indirect taxes		139.486		139.486
Total revenue		509.801		471.782
Expenditures				
Public consumption	243.905		243.905	
Public investments	19.501		19.501	
Public net interests expenditures	28.464		41.224	
Public net payments abroad	9.030		9.030	
Public pensions	101.439		79.831	
Public unemployment benefits	38.632		46.048	
Public age dependent transfers	38.427		82.641	
Public ordinary expenditures total		479.397		522.180
Other expenditures, net		0.000		-26.487
Public total lump sum transfers to households		30.403		0.000
Total expenditures		509.801		495.693
Budget surplus		0.000		-23.911

Sources: *ADAM's databank*, Statistic Denmark, *Nationalregnskab, offentlige finanser og betalingsbalance*, 1997:11, and own calculations.

Tabel 3. The public budget, bill. Dkr.

1. Other expenditures, net	-26.487
2. Total direct taxes	38.019
3. Net interest expenditures	12.760
4. Ordinary transfers to households	30.022
5. Budget surplus	-23.911
Sum = lump sum transfers to households	30.403

Table 4. Decomposition of the public lump sum transfers to households, bill. Dkr.

First, considering the model's level of abstraction, we do not find it worthwhile to take account of various smaller items of the public accounts. They are summed up in "other expenditures, net" which amount to a net revenue of 26.487 bill. Dkr.

Second, the revenue of total direct taxes are over-estimated by the model by 38.019 bill. Dkr. or 11.8 per cent of the base year revenue of 322.241 bill. Dkr. There is esveral reasons for this. Some of the direct tax rates are calculated such that they measure the average incentive impact on the agents and not such that they result in the actual revenue. Also, the model calculated tax bases might deviate from the actual tax bases for most direct taxes for various reasons, including that the composition of the benchmark population deviates from the actual population, the unrealistic assumption that all individuals who are between 18 and 60 years old belong to the labor force and the fact that all firms are assumed to be organized as corporations. Contrary to this, the indirect tax rates are calibrated as the correct revenue divided by the tax base, resulting in correct revenues for each of these taxes, cf. also chapter 7. The calculation of the direct tax rates for the base year is documented in Knudsen, Pedersen, Petersen, Stephensen & Trier (1997). The quantitative outcome is listed in table 7.1 of chapter 7.

Third, net interest expenditures are underestimated by 12.760 bill. Dkr. The implicit interest rate on the public debt is significantly larger than the interest rate of 5 per cent assumed in the model. This is because inflation compensations and risk premiums of the real world are not incorporated in the model. For simulations with the model a (real) rate of interest of 5 per cent should still be considered relevant for marginal experiments, while the difference in levels at the base year are compensated for by lump sum transfers.

Fourth, ordinary transfers to households are underestimated by 30.022 bill. Dkr. The main reason for this is that age dependent transfers has been underestimated by 44.214 bill. Dkr. because it has not been possible to to distribute all relevant transfers at age and gender and because the benchmark population deviates from the base year population. There are three main categories of transfers to households in the model.

- Pensions are defined as the sum of the general retirement pension inclusive supplements, civil servants' pensions, personal supplemnets, special pensions and early retirement pensions ("efterløn") which in 1995 amounted to 79.831 bill. Dkr. In the benchmark data set we further include health related early retirement pensions ("førtidspension") of 8.081 bill. Dkr. given to individuals above the model defined retirement age of 61. The average retirement age was

61 in the base year, but in the model it is simply assumed that all individuals above the age of 61 are pensioners and therefore they can not recieve health related early retirement pensions. The amount of retirement pensions is then $79.831 + 8.081 = 87.912$ bill. Dkr. There were 987,617 inhabitants at least 61 years old, giving an average pension of $87.912/987.617 = 0.089014$ million Dkr. The number of stationary state pensioners in the benchmark dataset is however 1139,488, giving total benchmark pension expenditures of 101.439 bill. Dkr.

- Base year unemployment benefits are defined as the sum of ordinary unemployment insurance payments and all social security benefits ("kontanthjælp") although the latter are not alloted to unemployed only. Therefore it is not worrying that the benchmark amount of 38.632 bill. Dkr. undershoots the base year amount 7.416 bill. Dkr. The benchmark amount hinges on the assumed replacement ratio, b_{t_0}, of 68 per cent which is documented in Knudsen, Pedersen, Petersen, Stephensen & Trier (1997).

- Age dependent transfers are defined as the remaining transfers which it is possible to distribute over sex and gender by means of the Danish generational accounting of Jacobsen *et al.* (1997). They consist of health related early retirement to individuals of age 18-60 years, and allowances to children, housing, education and sickness and childbirth leave. The undistributed remaining transfers are in effect assumed to be uniformly distributed on all adults of age 18-60 *via* the lump sum transfers.

Finally table 4 shows that the budget surplus of course contributes to the lump sum transfer.

Table 5 shows the net public debt.

	1 year value	5 year value
1. Central government net debt	578.300	115.660
2. Central bank own capital	36.227	7.245
3. Local government net debt	27.200	5.440
4. Total net public debt (1-2+3)	569.273	113.855

Note: Central government net debt includes assets at The Central Bank and The Social Pension Fund.

Sources: *Statens låntagning og gæld*, Nationalbanken 1996, p. 56.
Beretning og Regnskab, Nationalbanken 1996, p. 109.
Kvartalsoversigt, Nationalbanken 1997 / maj, table p. 15.

Table 5. The net public debt, bill. Dkr.

9.5.5 Current account

Table 6 shows the current account figures for 1995. Net exports, net interest reciepts and net public transfers from abroad are compiled according to national account definitions, cf. also table 1 and table 3, which deviates from balance of payments statistics.[9] The current account is defined according to the balance of payments statistics. In the base year data set, the difference between these two sources ends up in net transfers to households from abroad which are calculated residually. In the benchmark data set the current account surplus is assumed to be zero by further manipulating the net transfers to households from abroad.

	Benchmark	Base year
Exports	306970	306970
Imports	265573	265573
Balance on goods and services	41397	41397
Net interest reciepts	-13300	-26531
Net public transfers from abroad	-9030	-9030
Net transfers to households from abroad	-19067	4973
Current account	0	10809

Table 6. The current account, mill Dkr.

The net interest reciepts from abroad in the benchmark data deviates from the base year figure. This is analogous to the public sector net interest expenditures deviation discussed above.

9.5.6 Corporate debt share[10]

Jessen and Frederiksen (1991) estimated that for qouted Danish corporations, the share of own financing in the total balance was around 35 per cent in 1989. However, this figure represents a probably cautious and conservative accounting estimate where assets are valuated at replacement prices. In DREAM, corporate own capital should be assessed by its market value based on the expected future earnings, which is propably larger than the cautious accounting estimate. Therefore the own financing share is probably somewhat higher than the estimate of Jessen and Frederiksen. We assume that it is equal to 40 per cent giving a debt financing share at 60 per cent. OCED (1991) found that the debt financing of real investments is 65 per cent in Denmark.

[9] Total imports of table 1 include a tariff revenue of 1.936 bill. Dkr. which is subtracted from the import expenditures of table 6.

[10] This subsection reproduces the work of Schultz Møller (1993).

9.5.7 The wealth of households

Table 6 shows the benchmark calculation of households' total assets. The benchmark value of the firm is calulated from the benchmark dividend as described in chapter 7.

	1 year value	5 year value
1. (Net) Capital stock	2506.866	501.373
1.a. - of which foreign capital (60 %)	1504.120	300.824
2. Value of firm	1380.483	276.097
3. Public debt	569.273	113.855
4. Net foreign assets	-266.000	-53.200
5. Total household net wealth (1.a-4)	3187.876	637.575

Table 6. The benchmark wealth of households, bill Dkr.

9.5.8 Elasticities of substitution

The econometrically estimated (compensated) import price elasticities of the Danish macroeconometric model SMEC are converted to elasticities of substitution using the ordinary relation between the elasticity of substitution and the compensated own price elasticity

$$\sigma_{DF} = -e^k_{FF}/S_D \tag{9.33}$$

where σ_{DF} is the elasticity of substitution between the domestic and the foreign input, e^k_{DD} is the compensated own price elasticity of imports and S_D designates the domestic share of overall input costs (domestic plus imported). Table 7 illustrates how (9.33) is utilized to calculate the elasticities of substition for imports based on the import price elasticities of SMEC. The domestic market shares are calculated from the input output table, cf. table 1, as the share of the delivery from domestic production in the total delivery (domestic plus imports) to the demand category in question. It should be noted that the disaggregation of imports in SMEC is only compatible with that of DREAM as a rough approximation.

		Private sector		Public sector	
	(1)	(2)	(3)=(1)/(2)	(4)	(5)=(1)/(4)
	Import price elasticity	Domestic market share	Elasticity of substitution	Domestic market share	Elasticity of substitution
Materials inputs	-0.9	0.75	1.2	0.87	1.0
Private consumption	-1.2	0.79	1.5		
Investments	-1.2	0.79	1.5	0.91	1.3

Source: The import price elasticities are based on SMEC, cf. Det økonomiske Råd (1994).

Table 7. Calculation of the import elasticities of substitution

In ADAM the long run uncompensated own price elasticity of services in private consumption is estimated to -1.06, cf. Dam (1996). Most of the governmetally produced good in private consumption consists of services. We therefore assume that the own price elasticity of the governmentally produced good in private consumption is -1.06. The elasticity of substitution between the private and the governmentally produced good in private consumption, σ_{GP}, is calculated using the ordinary formula

$$\sigma_{GP} = \frac{e_{GG} - S_G}{S_G - 1} = \frac{-1.06 - 0.045}{0.045 - 1} = 1.1$$

where e_{GG} is the uncompensated own price elasticity of the governmentally produced good and S_G is the cost share of that good which is calculated from table 1.

Chapter 10

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