DREAM
Danish Research institute for
Economic Analysis and Modelling

MAKRO

MAKRO Model Documentation

Martin Bonde, João Ejarque, Grane Høegh,
Peter Stephensen, Tamás Vasi, Adam Mathiesen

June 2026

MAKRO Model Documentation

June 2026

Author(s):
Martin Bonde
João Ejarque
Grane Høegh
Peter Stephensen
Tamás Vasi
Adam Mathiesen

The report is available for download at
www.dreamgruppen.dk

ISBN web: 978-87-91638-13-8

Foreword

MAKRO is a large, empirically based macroeconomic model of the Danish economy. The purpose of the model is to serve as a tool for medium and long-term projections, short and long-run policy impact assessment, and long-run fiscal policy sustainability measurement. In addition, the model can incorporate short-term economic forecasts into its projections.

The model is developed by the MAKRO model group at DREAM (Danish Research Institute for Economic Analysis and Modelling) for use by the Danish Ministry of Finance. The model code is made freely available, however, and may be used by anyone.

The model parameters, equations, and data as a whole have been selected such that the short and long-run properties are as empirically and theoretically well-founded as possible. MAKRO incorporates four types of economic agents: households, firms, the government, and foreign agents demanding Danish exports. The model is micro-founded and forward-looking, with the behavior of households and firms supported by a detailed specification of their objective functions, preferences, technology, and budget sets. The government's behavior follows a set of exogenous rules as is standard, and the demand for exports is determined by a demand function that incorporates aspects of different trade models.

MAKRO is a New-Keynesian model, but differs from so-called DSGE models in a few ways. DSGE models solve for optimal decisions, which are functions of state variables and contain information pertaining to the model's stochastic nature. These optimal decision functions are defined in a neighborhood of the model's steady state. MAKRO is instead a computational general equilibrium model that solves for a single path for all its variables. This solution relies on initial and terminal conditions and reflects all policy changes and variations in exogenous factors one wishes to study.

MAKRO also differs from other models of its type due to its size. The household side of the model solves a model of overlapping generations, each with a life cycle of 100 years. The firm side of the model currently solves for nine different sectors of the economy. It is a nonlinear model with a large number of endogenous variables, and as a professional planning and budgeting tool, the model's variables must correspond precisely to their counterparts in the data. One of the primary purposes of MAKRO is to characterize the government budget balance and how it responds to shocks and policy changes. This requires a considerable disaggregation of the fiscal part of the model as it must be able to evaluate a large number of tax and transfer interventions as well as specific government consumption changes.

Contents

1 Introduction **10**

1.1 Computational MAKRO 13

1.1.1 Code organization 13

1.1.2 Notation 14

2 Households **15**

2.1 Basic definitions 15

2.2 The household problem 15

2.3 Optimization 16

2.3.1 Savings decision 16

2.3.2 Housing decision 17

2.3.3 Putting the two together 18

2.4 Children 19

2.5 Migration 19

2.6 Detailing the f housing object 20

2.7 Household Income 21

2.7.1 Income 22

2.8 Pensions 22

2.8.1 Pension Contributions 23

2.8.2 Pension wealth 23

2.8.3 Pension Income 23

2.8.4 Composition and returns of pension portfolio 24

2.9 Bequests 24

2.9.1 Death 24

2.9.2 Bequests received, liquidating housing, and bequests in utility 24

2.10 Household financial portfolio 25

2.11 Utility 26

2.11.1 Utility of consumption and housing 26

2.11.2 Habit formation and reference consumption 26

2.11.3 Utility of Bequests . . . 28
2.11.4 Utility of Wealth . . . 29
2.11.5 Disutility of house price changes. . . . 29
2.11.6 Differentiating with respect to housing . . . 30
2.12 Consumption components . . . 31
2.12.1 The nested consumption components . . . 31
2.12.2 CES optimization . . . 32
2.12.3 Tourism . . . 33
2.13 Appendices . . . 34
2.13.1 Calculating the bequest allocation matrix . . . 34
2.13.2 Estimating assets and Liabilities as functions of B . . . 36
2.13.3 Land and housing depreciation . . . 42
2.13.4 Mortgages and Housing in the budget constraint . . . 43
2.13.5 The Mortgage Ratio μ. . . . 45
2.13.6 Aggregation . . . 46
2.13.7 Housing intermediary . . . 49
2.13.8 Merging profits of intermediary with the household budget constraint. . . . 50
2.13.9 The multiplier effects of leverage. . . . 51

3 Firms **53**
3.1 Cost minimization . . . 54
3.1.1 The production function . . . 54
3.1.2 The CES tree . . . 54
3.1.3 CES cost minimization . . . 54
3.2 Dynamic optimization . . . 55
3.2.1 Definitions . . . 55
3.2.2 Debt . . . 56
3.2.3 The problem of the firm . . . 57
3.2.4 First-order conditions . . . 58
3.2.5 Factor utilization . . . 59
3.3 Firm investments with costly external finance . . . 60
3.3.1 An illustrative example . . . 60

3.3.2 The financial constraint in MAKRO 62
3.3.3 Homogeneity 62
3.3.4 Acceleration 63
3.4 Finance 63
3.4.1 Valuation algebra 64
3.4.2 Financial accounting terms and corporate taxes 65
3.5 Special sector considerations 66
3.5.1 The extraction sector 66
3.5.2 The housing sector 66

4 Price setting **69**
4.1 Monopolistic competition and price rigidity 69
4.2 Monopolistic Competition Model 70
4.2.1 Optimization Problem 70
4.3 Sector considerations 72
4.4 Foreign prices 72

5 The input/output system **74**
5.1 Market clearing prices 74
5.2 Demand trees 76
5.2.1 Aggregation of foreign and domestic inputs 77
5.2.2 Aggregation of quantities from different sectors 79
5.2.3 Exports 79
5.3 Aggregation 79
5.4 Appendices 80
5.4.1 Details on investments 80
5.4.2 Data and calibration 81
5.4.3 Balancing share parameters 82

6 Exports **84**
6.1 Export of domestically produced goods 84
6.1.1 Demand for the five export groups 84
6.1.2 Market size 85

6.1.3 Price ratio . . . 85
6.1.4 Export elasticity . . . 86
6.1.5 Scale . . . 86
6.1.6 Energy exports . . . 86
6.1.7 Tourism divided on consumption groups . . . 87
6.2 Imports for export . . . 87
6.3 Derivation of the dynamic price ratio . . . 89

7 Labor market 91
7.1 Household problem . . . 91
7.1.1 Household problem . . . 92
7.1.2 Optimal choice of hours . . . 94
7.1.3 Optimal choice of labor market participation . . . 94
7.2 Non-resident workers . . . 98
7.3 Firms . . . 98
7.3.1 The problem of the firm . . . 99
7.3.2 Wages and production input . . . 100
7.3.3 Vacancy posting costs . . . 100
7.3.4 Detailing the first-order condition . . . 101
7.3.5 Link to CES optimization . . . 102
7.4 The matching function . . . 102
7.5 Aggregation . . . 103
7.5.1 Hours and productivity . . . 103
7.5.2 Employment law of motion, firm's point of view . . . 104
7.6 Wage determination . . . 105
7.6.1 Bargaining . . . 105
7.7 Appendices . . . 107
7.7.1 Aggregating household employment with migration . . . 108
7.7.2 Imposing the same age distribution on every firm . . . 109
7.7.3 Different average wages across sectors . . . 109
7.7.4 Labor market variables . . . 110

8 Structural employment and output 114

8.1 Structural employment . . . 114
8.1.1 Law of motion of structural employment . . . 115
8.1.2 Non-resident workers . . . 115
8.1.3 Structural choice of labor market participation . . . 116
8.1.4 Structural choice of hours . . . 116
8.1.5 Sector composition . . . 117
8.1.6 Structural labor demand . . . 117
8.1.7 Structural wage . . . 118
8.2 Structural output (gross value added) . . . 118

9 Public production **120**
9.1 Supply side . . . 120
9.1.1 Input method . . . 122
9.1.2 Public employment . . . 122
9.1.3 Public investments . . . 123
9.1.4 Intermediate input purchases . . . 124
9.2 Demand for public production . . . 124
9.2.1 Private consumption of public services and public direct investments . . . 125
9.2.2 Government consumption of public production . . . 125
9.2.3 Baseline government consumption . . . 126
9.3 Appendices - Public Production . . . 127
9.3.1 Productivity Growth . . . 127

10 Government **129**
10.1 Revenue . . . 129
10.1.1 Direct taxes . . . 130
10.1.2 Income terms and allowances . . . 133
10.1.3 Indirect taxation . . . 135
10.1.4 Other government revenues . . . 137
10.2 Expenditures . . . 138
10.2.1 Income transfers . . . 139
10.2.2 Age distribution of income transfers . . . 140
10.2.3 Subsidies . . . 142

10.2.4 Other expenses 143

10.3 Net interest income and government net wealth 143

10.4 The fiscal sustainability indicator 144

11 References **146**

1 Introduction

The documentation contained in the subsequent chapters is a description of the model version 2024-December. Although it is written mainly for model users to understand the background of the computational code, each chapter contains a description of the relevant theoretical part that a wider audience can understand. The following paragraphs summarize the content of each chapter.

Households. This chapter lays out the household problem.

Optimizing households solve a dynamic life cycle problem within an overlapping generations model. They maximize utility by choosing optimal non-durable consumption and savings into liquid assets, optimal housing, optimal job search effort, and hours worked. Within non-durable consumption, they also decide on the optimal composition of a consumption bundle. The consumption/savings decision is dynamic and forward-looking. Households choose the total amount of liquid non-housing net financial assets. We allocate the total volume of wealth to different assets in a portfolio composition estimated from the data. The optimal housing decision is also dynamic and forward-looking and reflects costs of mortgage financing, housing depreciation and maintenance, and capital gains from house prices. The optimal choice of the non-durable consumption bundle is a static decision organized in a sequence of cost-minimization problems. The optimal job search decision is also a dynamic, forward-looking decision. Bells and whistles are added to the utility function so as to be consistent with empirical data. We add reference consumption in utility for both durable and non-durable consumption. More precisely, we add habits in the utility function so as to be consistent with hump-shaped responses as typically observed in macro-data.

Household members die in our model, leaving bequests associated with warm glow utility when they die. Bequests received are taken as given by the optimizing agent. The mapping between bequests given and received at different ages is an allocation matrix estimated from the data, which enters the model exogenously.

Firms. This chapter details production. Domestic output is produced by private firms and by the government. The economy has eight private production sectors corresponding to eight broad classes of goods and services. Firms use capital, labor, and intermediate inputs in each sector to produce output. Quantities of inputs are combined in a production function to produce units of output. Capital is subject to a time-to-build constraint of one period which makes investment decisions forward-looking, and to investment adjustment costs which makes the optimal decision dynamic. Capital goods can be purchased from multiple supplying sectors and both domestic and foreign sources. Employment adjustment is also forward-looking and subject to frictions. Firms incur costs from posting vacancies that are filled with a probability that is outside the firm's control. The optimal use of intermediate inputs is a static decision, and, like capital goods, these can also be purchased from multiple supplying sectors and domestic and foreign sources. Firms are price takers in input markets. Public production differs from its private counterpart and is detailed in the chapter on public production.

Price setting This chapter explains how prices are determined in the economy. The problem faced by firms is twofold: cost optimization and setting the optimal output price. The previous chapter deals with cost optimization, while this chapter focuses on setting the optimal output price. In a perfect competition setting, the optimal output price would equal the unit costs. However, observed prices are more sluggish than the ones generated by the perfect competition solution, which leads to the addition of an intermediate layer of price-setting behavior between the producing firm and the agents demanding the output.

This intermediate layer is modeled as a monopolistic competition problem, which generates price rigidity. In this setup, all firms within each sector face the same demand elasticity, and the aggregate price over all firms in a given sector is a CES price index. Price stickiness originates from quadratic adjustment costs.

Input/output structure. The input/output chapter details the collection of market clearing conditions, where the demand for intermediate inputs, private consumption, government consumption, investment, and exports is met by supply from domestic and foreign producers. The supply side of the IO structure consists of nine domestic and four import sectors. The demand side ultimately disaggregates to the same nine sectors, however, demanded goods have heterogeneous degrees of intermediate aggregation. Intermediate inputs into production have different compositions for each of the nine production sectors. Consumption is split between 5 private consumption groups and a single type of government consumption. Investments are in structures, equipment, or inventories, and finally, demand for exports distinguishes between five different export groups (including tourism exports which are included in the private consumption groups).

These mappings, for example, between the definition of the five consumption goods demanded by households and the nine different production sectors, can, on the one hand, be viewed as demand coming directly from households and into the various production sectors through layers of nested sub-utility functions. But, on the other hand, they can also be viewed as layers of zero-profit markets/firms that transform the basic goods into the upper-level goods the agents demand. This transformation occurs via constant returns to scale "technology", which generates the necessary prices.

Finally, at the very bottom of the demand side construction is the decomposition between domestic production and imports by a constant elasticity of substitution aggregator. At this level, there is substitutability between domestic and foreign supply in response to price changes.

Exports. Exports are modeled using a reduced form incorporating insights from various trade models and mirror the determinants of imports generated by MAKRO. The Export demand equation includes the following:

- A measure of the size of the export market.
- A price ratio measure of our competitiveness in this market.

- A measure of domestic output.
- Lagged exports.

The price elasticities of export demand in the different exported goods are fundamental parameters in MAKRO, as in any small open economy model. They are a key source of concavity in an otherwise largely linear model and allow the model to have a finite solution. Therefore, significant effort and care have been taken in the econometric estimation of these parameters. Details of the econometric work are available in additional documentation.

Labor Market. The labor market includes heterogeneous households and firms. Households make decisions about their labor market participation and working hours, while firms post job vacancies, and a matching function brings workers and vacancies together. The model includes bargaining between agents representing workers and firms to set the market wage. The household side is age-specific, and the firm side is sector-specific.

Structural Levels. This chapter discusses how we calculate structural employment and output, which are essential for calculating the structural budget balance.

Structural employment is based on a modified labor market framework where we remove nominal frictions and rigidity stemming from matching frictions. Structural employment is thereby determined by the structural labor force and the institutional structures of the labor market. The model can, in general, provide conditional forecasts of the effects of shocks or policy interventions. However, MAKRO does not unconditionally forecast the structural labor force or hours worked. Instead, these variables are treated as exogenous inputs based on demographic projections.

From structural employment, we calculate structural output through a production function that uses the actual stock of capital but with neutral utilization rates.

Public production. This chapter describes how public sector output, comprised of goods and services, is provided by the state and consumed domestically. Public production is produced just as private sector goods in the sense that it uses labor, capital equipment and structures, and intermediate inputs. The default setting in MAKRO is that all factors of public production (labor, capital equipment and structures, and intermediate inputs) are exogenous. The quantity of output produced is endogenous, but only reacts to shocks to the production factors. Most public services are not traded on any market, and therefore no market prices exist in the data. The accounting price of public production is instead the average cost of production. Rather than the production function approach used for private sector production, we use an input method, where chain indices are used to distinguish between changes in public sector prices and quantities.

Government. One essential purpose of MAKRO is to determine the government budget balance and how it responds to shocks and policy changes. The government chapter provides a breakdown of government revenues and expenditures and how each revenue or expenditure item is modeled. The fiscal sustainability indicator (HBI) summarizes the overall long-run state of public finances, and we show how it is calculated.

Calibration and Estimation. The document "The empirical basis for MAKRO" describes our overall empirical strategy. In addition, we publish working papers on the DREAM website with detailed descriptions of different econometric procedures and empirical results.

Most parameters are calibrated using available data (over 1,500 in the latest version) such that the model is consistent with the national accounts. Most of these are "level parameters," such as the scale parameters in CES functions, which ensure that MAKRO hits the right levels for the data-covered endogenous variables. Most calibrated parameters are determined using a single static relation/equation. Solving these parameters using data comprises our "static calibration" procedure, yielding time series of these parameters for the available historical data period. Other parameters are determined using dynamic relationships such as forward-looking first-order conditions. These parameters are calibrated in our "dynamic calibration" procedure. Before performing dynamic calibration, we need to forecast some parameters obtained in static calibration.

The static calibration process generates historical time series for the different parameters. These time series can display structural trends, such as a growing service sector, and capture short-run fluctuations and structural breaks. This information is treated econometrically with ARIMA models to generate forecasts of parameter values. With these in hand, we can solve the forward-looking equations to recover the associated parameters.

Finally, some parameters are closely related to short-run fluctuation behavior. These parameters are estimated by shocking the model and comparing the resulting impulse responses in artificial data with those obtained from SVAR models and other empirical approaches. This is a standard methodology in the DSGE literature, which we bring into our CGE framework.

1.1 Computational MAKRO

MAKRO is coded in GAMS which is an efficient software for solving large scale systems of nonlinear equations.

1.1.1 Code organization

The code is organized in modules reflecting the theoretical chapters in this documentation. Each module can be solved separately but requires inputs from and provides outputs to other modules.

The code modules are: *Consumers* and *Household Income*, *Finance* and *Private production*, *Pricing*, *Input/Output*, *Exports*, *Labor market*, *Structural levels*, *Public production*, *Government*, *Government expenses* and *Government revenues*, *Taxes*, and *Aggregates*.

1.1.2 Notation

One problem that arose was that of having a system to name the large number of variables and parameters in the model. Notation in the documentation is consistent with the code but not identical. Nearly all objects in the code have long descriptive names that allow for their identification in a dense computational environment. The code names are mainly in Danish because the users of this code will be Danish, while in the documentation, the working language is English, as the model is meant to be understood by a universal audience.

Some simple organizational choices are made for names in the code: quantities have prefix q, prices have prefix p and nominal values have prefix v. Many variables are recognizable in the code using common sense: K is associated with capital, L with labor, C with consumption, Y with output, etc.

In the documentation, most object names are much shorter to ease notation, while Greek letters are used for parameters following the standard practices of the academic literature. As an example, a depreciation rate will be labeled \delta in the documentation while having a long name in the code. One Greek letter pervasive in the documentation is μ. This character typically denotes share parameters which are a part of the widely used CES tree approach in production and consumption, and in the code the letter u replace it. While in the documentation, μ will be used identically in different chapters without risk of confusion, in the code, u will have additional characters and indices added to provide identification.

Another aspect of variable name organization is naming the same object at different levels of aggregation. This can be done by extending the variable name, for example, to aggregate or consider an age-specific quantity or by using the same name with additional indexing. For example, a superset s* can contain not just the nine items pertaining to the nine different production sectors in s, but also different subsets of the elements in the set s, allowing for various degrees of aggregation without changing the name of a variable.

A critical aspect of the code, and one crucial capability of GAMS is organizing the data using indices and sets. As the model has many demand and supply side items, identifying such items occurs through appropriate set description and indexing. For example, an object such as q[d,s,t] will denote the quantity q demanded by sector d and supplied by sector s at time t.

The most important sets are time (t), which runs from 1965 to 2099; age (a), which runs from 0 to 100, and the non-numerical set, s, which currently has nine values identifying eight private sectors and one public sector. Additional sets are used to index capital goods, consumption goods, export goods, and intermediate inputs. Of these, the last three sets (consumption (c), exports (x), intermediate inputs (r)) are demand-side reorganizations of the nine sector production set s. The index for capital goods covers equipment, structures, and inventories and is independent.

2 Households

The problem of the household is to choose optimal amounts of savings and expenditure, and within the expenditure, choose the different types of goods they consume. A particularly important good is housing as the model must replicate several important features of the data. These features include moments at aggregate level and life cycle profiles of housing ownership, mortgage debt, non-housing wealth, non-housing consumption, and the intertemporal marginal propensity to consume out of different shocks.

The household problem contains several building blocks and we start by presenting the baseline case first. Afterwards, we outline in detail how household income is defined, how the financial portfolio is constructed, how the optimal decomposition of non-housing consumption occurs, and how bequests are allocated.

2.1 Basic definitions

The model is a discrete-time, perfect foresight, overlapping generations model of the life cycle. The size of the cohort aged a in period t is $N_{a,t}$ and is exogenous. The household problem is to choose an optimal consumption path over the life cycle given its income path. The income path is endogenous as the household decides also on its participation in the labor market. However, as labor disutility is separable from other utility, we can treat this choice in isolation and it is thus discussed in the labor market chapter. Furthermore, consumption of different non-housing goods is the result of a CES nest optimization sequence which relates to the input/output structure of the data, detailed in 2.12.

Henceforth we use the following symbols with the associated purposes: η will denote an elasticity, δ a destruction or depreciation rate, τ will denote a tax rate, and β will be the household's preference discount factor.

2.2 The household problem

Households of age a in period t make consumption and savings decisions to solve the following Bellman equation[1]

$$\begin{aligned} V_{a,t}\left(B_{a-1,t-1}, D_{a-1,t-1}\right) = & \max_{C_{a,t}, D_{a,t}} U\left(C_{a,t}, D_{a,t}\right) \\ & + \beta V_{a,t}^{Wealth}\left(B_{a,t}, D_{a,t}\right) \\ & + \beta \mathbb{W}_{a+1,t+1}\left(B_{a,t}, D_{a,t}\right) \end{aligned} \tag{2.1}$$

s.t.

$$B_{a,t} = \left(1 + r_{a,t}\right) B_{a-1,t-1} + y_{a,t} - p_t^c C_{a,t} - f\left(D_{a,t}, D_{a-1,t-1}\right)$$

[1] For $a \geqq 18$. Consumption for children under the age of 18 is assumed to be part of their parents' consumption.

where $a = 0, 1, 2, ..., A$ indexes age with the index value $a = 0$ belonging to the first age of life when children are born and until they become one year old. The letter t indexes time.

In the budget constraint $y_{a,t}$ denotes income excluding returns on liquid assets, $B_{a,t}$ is the net stock of liquid assets, p_t^c is the price of non-housing consumption $C_{a,t}$, and $f(\cdot)$ is a function capturing all elements of the budget constraint related to owned housing, (described in detail in section 2.6). Non-housing consumption, $C_{a,t}$, is split into several different consumption goods according to a CES tree structure, see Section 2.12.

In the objective function $U(\cdot)$ is instantaneous utility over non-housing consumption, $C_{a,t}$, and housing $D_{a,t}$, both of which have a habit component. This habit is external in the case of non-housing consumption, and internalized in the case of housing. The object $V_{a,t}^{Wealth}(\cdot)$ is the function capturing direct utility of wealth, and $\mathbb{W}_{a+1,t+1}(\cdot)$ is the continuation-value of the problem. This continuation value contains a probabilistic element in the form of the age-specific survival probabilities, s_a, such that $\mathbb{W}_{a+1,t+1}(\cdot)$ is given by

$$\begin{aligned}\mathbb{W}_{a+1,t+1}\left(B_{a,t}, D_{a,t}\right) = {} & s_a V_{a+1,t+1}\left(B_{a,t}, D_{a,t}\right) \\ & + \left(1 - s_a\right) V_{a+1,t+1}^{Beq}\left(B_{a,t}, D_{a,t}\right)\end{aligned} \tag{2.2}$$

where $V^{Beq}(\cdot)$ is utility of leaving bequests in the case of not surviving till the next age of life.

Finally, the Bellman problem has household-specific *state variables* which are the beginning of period stock of liquid wealth, $B_{a-1,t-1}$, the stock of housing, $D_{a-1.t-1}$, and the household age, a.

2.3 Optimization

2.3.1 Savings decision

The first-order condition for savings can be obtained by replacing the non-housing consumption variable with the budget constraint in the Bellman equation, eq. (2.1), and choosing end of period stock of assets $B_{a,t}$. We obtain

$$\frac{U_{a,t}^c}{p_t^c} = \beta \left\{ s_{a,t}\left(1 + \overline{r}_{t+1}^B\right) \frac{U_{a+1,t+1}^c}{p_{t+1}^c} + s_{a,t} \frac{\partial V_{a,t}^{Wealth}}{\partial B_{a,t}} + \left(1 - s_{a,t}\right) \frac{\partial V_{a,t}^{Beq}}{\partial B_{a,t}} \right\} \tag{2.3}$$

where $\overline{r}_{t+1}^B$ is the marginal rate of return,

$$\overline{r}_{t+1}^B = \frac{\partial\left(r_{t+1} B_{a,t}\right)}{\partial B_{a,t}}$$

Equation (2.3) is a standard Euler equation, wherein the household trades-off current with future marginal utility of consumption. On the left hand side, the last unit of income used for current consumption yields $1/p_t^c$ units of consumption with marginal utility $U_{a,t}^c$. Optimality implies this must be identical to what is obtained from alternatively saving this marginal unit of income, earning a net marginal return $\overline{r}^B_{t+1}$, and using it next period for consumption, taking into account that one may die. This is given by

$$\left(1+\overline{r}^B_{t+1}\right)\frac{U^c_{a+1,t+1}}{p^c_{t+1}}$$

weighed by the survival rate $s_{a,t}$ and discounted by the factor β to match the current marginal utility. In addition, a surviving household will also derive the utility of the ownership of the extra wealth. On the other hand, in the chance $(1-s_{a,t})$ that a household does not survive, it obtains the marginal change in bequest utility, $\partial V^{Beq}_{a,t}/\partial B_{a,t}$, measured in the future and discounted back for mechanical consistency, as in case of death the agent only dies tomorrow (and therefore after the current savings decision).

Terminal savings condition (last period of life) The household lives up to 100 years of age. The survival rate is thus zero in the last age, $s_{A,t}=0$, but bequests still occur. We obtain

$$\frac{U^c_{A,t}}{p^c_t}=\beta\frac{\partial V^{Beq}_{A,t}}{\partial B_{A,t}},$$

which determines assets at the end of life. However, setting the survival rate at zero induces an abrupt change in behavior at the end of life that distorts the optimal choice due to the truncation of life. We instead use the following equation where the survival rate is the actual rate observed at 100 years of age, $s_{A,t}\neq 0$, and where we replace the would-be consumption of 101-year-olds with the consumption of next period's 100-year-olds:

$$\frac{U^c_{A,t}}{p^c_t}=\beta\left\{s_{A,t}\left(1+\overline{r}^B_{t+1}\right)\frac{U^c_{A,t+1}}{p^c_{t+1}}+s_{A,t}\frac{\partial V^{Wealth}_{A,t}}{\partial B_{A,t}}+\left(1-s_{A,t}\right)\frac{\partial V^{Beq}_{A,t}}{\partial B_{A,t}}\right\}$$

2.3.2 Housing decision

Housing $D_{a,t}$ is a stock variable. Like the choice of $B_{a,t}$, the choice of housing is a dynamic forward-looking decision with an associated intertemporal first-order condition. The general expression for this condition is[2]

$$\frac{U^c_{a,t}}{p^c_t}\left(\frac{\partial f_{a,t}}{\partial D_{a,t}}\right)=U^d_{a,t}+\beta s_{a,t}\frac{\partial U_{a+1,t+1}}{\partial D_{a,t}}+\beta s_{a,t}\frac{U^c_{a+1,t+1}}{p^c_{t+1}}\left(\overline{r}^D_{t+1}-\frac{\partial f_{a+1,t+1}}{\partial D_{a,t}}\right)$$
$$+\beta s_{a,t}\frac{\partial V^{Wealth}_{a,t}}{\partial D_{a,t}}+\beta\left(1-s_{a,t}\right)\frac{\partial V^{Beq}_{a,t}}{\partial D_{a,t}} \tag{2.4}$$

with

$$\overline{r}^D_{t+1}=\frac{\partial\left(1+r_{t+1}\right)B_{a,t}}{\partial D_{a,t}}$$

and where the presence of $\frac{\partial U_{a+1,t+1}}{\partial D_{a,t}}$ reflects the fact that the housing habit is internalized. This condition reads: when you sacrifice $1/p^c_t$ units of non-durable consumption today and use the money to buy extra housing, there is an immediate marginal utility loss from reduced

[2] Just as in the savings optimal choice, this equation needs to be adjusted in the final age of life.

consumption. This is the left-hand side of the equation. On the right-hand side, you gain the direct marginal utility of the durable good, $U^d_{a,t}$, and tomorrow you gain the marginal utility of non-durable consumption associated with the effect of the additional housing bought today on tomorrow's income, conditional on surviving. The direct effect of housing on tomorrow's budget is $-\frac{\partial f_{a+1,t+1}}{\partial D_{a,t}}$. Housing decisions may also affect tomorrow's budget through portfolio choices that change the rate of return on B, which is captured by $\overline{r}^D_{t+1}$, see section 2.10. In the case of not surviving, you instead gain bequest utility in period $t+1$.

2.3.3 Putting the two together

It is useful to combine the two first-order conditions, (2.3) and (2.4), to get an expression for the user-cost of housing. . Here we merge them by eliminating the marginal utility of consumption in period t+1, $U^c_{a+1,t+1}$. We obtain

$$\begin{aligned}\frac{U^c_{a,t}}{p^c_t}\left\{\frac{\partial f_{a,t}}{\partial D_{a,t}}+\frac{1}{1+\overline{r}^B_{t+1}}\frac{\partial f_{a+1,t+1}}{\partial D_{a,t}}-\frac{\overline{r}^D_{t+1}}{1+\overline{r}^B_{t+1}}\right\} &= U^d_{a,t}+\beta s_{a,t}\frac{\partial U_{a+1,t+1}}{\partial D_{a,t}}\\ &+\beta s_{a,t}\left[\frac{\partial V^{Wealth}_{a,t}}{\partial D_{a,t}}-\frac{\overline{r}^D_{t+1}-\frac{\partial f_{a+1,t+1}}{\partial D_{a,t}}}{1+\overline{r}^B_{t+1}}\frac{\partial V^{Wealth}_{a,t}}{\partial B_{a,t}}\right]\\ &+\beta\left(1-s_{a,t}\right)\left[\frac{\partial V^{Beq}_{a,t}}{\partial D_{a,t}}-\frac{\overline{r}^D_{t+1}-\frac{\partial f_{a+1,t+1}}{\partial D_{a,t}}}{1+\overline{r}^B_{t+1}}\frac{\partial V^{Beq}_{a,t}}{\partial B_{a,t}}\right]\end{aligned}$$

This yields the user-cost of housing expression measured at time t such that it is comparable to the current consumption price:

$$p^{ucD}_{a,t}=\underbrace{\left\{\frac{\partial f_{a,t}}{\partial D_{a,t}}+\frac{1}{1+\overline{r}^B_{t+1}}\frac{\partial f_{a+1,t+1}}{\partial D_{a,t}}-\frac{\overline{r}^D_{t+1}}{1+\overline{r}^B_{t+1}}\right\}}_{\text{User-cost of } D_{a,t} \text{ measured at time t in nominal units.}}$$

This merged equation has an intuitive reading. When the household sacrifices one unit of current consumption to buy additional housing, it must weigh the direct marginal utility of housing $U^d_{a,t}$ against the loss of marginal utility of consumption, $U^c_{a,t}$, net of the gain in the marginal utility of wealth and bequests.

We can also merge the two first-order conditions by eliminating the current marginal utility of consumption $U^c_{a,t}$. This is useful because, given the assumptions we make regarding utility of wealth and bequests[3], we can also eliminate marginal utility of wealth and bequests from the resulting condition:

$$U^d_{a,t}+\beta s_{a,t}\frac{\partial U_{a+1,t+1}}{\partial D_{a,t}}=\beta s_{a,t}p^{ucD}_{a,t}\frac{1+\overline{r}^B_{t+1}}{p^c_{t+1}}U^c_{a+1,t+1}$$

[3] Specifically we use that $\frac{\partial V^{Beq}_{a,t}}{\partial D_{a,t}}=\frac{\partial V^{Beq}_{a,t}}{\partial B_{a,t}}\frac{\partial f_{a,t}}{\partial D_{a,t}}$ and $\frac{\partial V^{Wealth}_{a,t}}{\partial D_{a,t}}=\frac{\partial V^{Wealth}_{a,t}}{\partial B_{a,t}}\frac{\partial f_{a,t}}{\partial D_{a,t}}$, see section 2.11.

The intuition here is also clear. An additional unit of housing today yields the corresponding direct marginal utility of housing (adjusted for habit). This must be identical to the marginal utility of consumption we could obtain tomorrow if instead of spending the money on housing we took that amount of money, $p^{ucD}_{a,t}$, capitalized it $\frac{1+\bar{r}^B_{t+1}}{p^c_{t+1}}$, and used it to eat in case we survive.

Given the behavioral relations of the households, we proceed to various details around age, death, and the details of the components of consumption and the budget constraint

2.4 Children

Our consumer starts life as a teenager. The data reveals income and assets for children below the optimizing age in the model, which is 18 years.[4] Fitting the budget constraint of these children is important, as it allows us to calibrate initial wealth at 18 years of age correctly and correct for otherwise excessive consumption of the associated parental household.

Rather than modeling children as optimizing agents, we let their consumption be implicit in the parent's problem and create an exogenous income transfer variable from parents to children that will fit the child's budget constraint at zero consumption and is just enough to hit the asset target at age 18. Children are born with zero assets, and for a few ages, they actually have recorded disposable income, so their budget constraints are given by

$$B_{0<a<18,t} = (1 + r_{a,t}) B_{a-1,t-1} + y_{a,t} + Transfer_{a,t}$$
$$B_{0,t} = 0$$

As we calibrate this equation to the data, we obtain the value of initial assets for agents at the first optimizing age. Note that for the purpose of this document, the transfer from the parent to the child is hidden inside the disposable income variable of the parent.

2.5 Migration

The population of a given age at a point in time will generally be such that

$$N_{a,t} = N_{a-1,t-1} s_{a-1,t-1} + I_{a,t} - E_{a,t}$$

where some agents will have either emigrated, $E_{a,t}$, or immigrated, $I_{a,t}$, the country at this point.

We make the necessary assumptions to ensure that those entering the country have the same consumption, income, housing, and employment as surviving residents; otherwise, the model would have an intractable amount of heterogeneity. Those leaving take their assets with them. As for housing, agents leaving sell their housing stock while agents entering come in with zero housing, such that the total amount of housing in the country is unchanged by immigration

[4] Optimal labor search decisions start at age 16, and this is possible because, by eliminating wealth effects from the labor decision we make the two problems independent of each other.

and retains its characteristic of being a good that is not traded across borders.

2.6 Detailing the f housing object

The housing the household buys and sells is an aggregate of "bricks" and land. The "bricks" part of the house is produced mainly with inputs purchased from the construction sector. The country's entire stock of land is held by households embodied in their housing good. Land available for the construction of new houses is the land released as a result of housing depreciation.[5] An intermediary buys output from the construction sector and other intermediate inputs, buys land from households released from depreciated housing, packages these together, and sells the resulting houses back to households. Land is introduced in MAKRO as a rigid production factor. In reality, land is a somewhat flexible factor, and we allow for exogenous increases in the aggregate stock.[6]

We introduce an exogenous supply of rental accommodation, H, including an exogenous rent, to capture the non-negligible amount of existing public and regulated rental housing. We do not model the link between the rental market and the owned housing market. Thus, rent expenses appear only as an exogenous term in the budget constraint of the household.

The costs and returns of housing Owned housing is the overwhelming source of household debt, and housing finance is a significant fraction of total financial activity. Houses here are financed with a mortgage with an age-specific fraction of mortgage financing to house value, $\mu_{a,t}$, as a loan-to-value (LTV) term. The object $\mu_{a,t}$ is exogenous to the household, but is modeled so that mortgages only gradually adjust to housing prices.[7] Therefore, the model generates quantities for mortgage debt that change via the extensive margin (as the stock of housing changes) and via movements in prices when the extensive margin is constant. The detailed modeling of $\mu_{a,t}$ is discussed in Appendix 2.13.4.

Housing enters the budget constraint via the object f. This object is a cost function which contains financing, taxation, and maintenance, and deducts revenues from downsizing and from land sales. We detail the elements of f with extended algebra and proofs in Appendix 2.13.4.

$$f\left(D_{a,t}, D_{a-1,t-1}\right) = \left(1-\mu_{a,t}\right) p_t^D D_{a,t} + \left\{\left(1+r_t^{mort}\right)\mu_{a-1,t-1} + \tau_t^W + x_t - \frac{p_t^D}{p_{t-1}^D}\left(1-\delta_t^d\right) - \alpha_t^{Land}\right\} p_{t-1}^D D_{a-1,t-1} \tag{2.5}$$

Since f is a cost function we have non-mortgage financing $(1-\mu_{a,t})$, property value taxes τ_t^W and maintenance costs x_t, and mortgage interest payments r_t^{mort}, all with a positive sign. Carried over un-depreciated housing, and income earned from selling land, are revenues and therefore appear with a negative sign. Appendix 2.13.3 details the computation of the factor α_t^{Land} which defines the revenue from land sales. The partial derivatives of expression (2.5),

5 Only the bricks part of the house dies with depreciation. The corresponding land is sold. The depreciation rate δ is detailed in the appendix, see Appendix 2.13.3.
6 Davis and Heathcote (2006), The Price and Quantity of Residential Land in the United States.
7 Endogenous mortgage ratios require the specification of mortgage contracts and a model of credit.

which enter the user cost expression and the optimality condition, are now trivial to compute and are all either exogenous to or taken as given by the household.

Hidden user costs. We incorporate an additional component in the user cost of housing to reflect the fact that houses have historically provided a higher but more volatile return than relatively risk free bond rates. As we cannot model stochastic returns while keeping the model tractable we instead add a risk premium directly (just as we do in the required return of equity). The term may also capture costs related to asymmetric information risk and other transaction costs. In our implementation, the risk premium enters as additional discounting of the expected future value of the house and the expected revenue from land sales. This means that the household applies a higher discount rate to the uncertain capital gain component of the housing return, while known obligations such as mortgage payments, property taxes, and maintenance costs remain discounted at the standard rate. The risk premium does not enter the budget constraint directly.

$$p^{ucD}_{a,t} = \frac{\partial f_{a,t}}{\partial D_{a,t}} + \frac{1}{1+\overline{r}^B_{t+1}} \frac{\partial \hat{f}_{a+1,t+1}}{\partial D_{a,t}} - \frac{\overline{r}^D_{t+1}}{1+\overline{r}^B_{t+1}} - \frac{\left(1-\delta^d_{t+1}\right) p^D_{t+1} + \alpha^{Land}_{t+1} p^D_t}{1+\overline{r}^B_{t+1} + \psi^{Risk}_{t+1}}$$

where $\partial \hat{f}/\partial D_{a,t}$ collects only the terms in $\partial f_{a+1,t+1}/\partial D_{a,t}$ that do not involve future prices or land revenues:

$$\frac{\partial \hat{f}_{a+1,t+1}}{\partial D_{a,t}} = \left\{\left(1+r^{mort}_{t+1}\right)\mu_t + \tau^W_{t+1} + x_{t+1}\right\} p^D_t$$

$$\frac{\partial f_{a,t}}{\partial D_{a,t}} = \left(1-\mu_{a,t}\right) p^D_t$$

Expanding and collecting terms, the user cost becomes:

$$p^{ucD}_{a,t} = \left(1-\mu_{a,t}\right) p^D_t - \frac{\overline{r}^D_{t+1}}{1+\overline{r}^B_{t+1}} + \frac{\left\{\left(1+r^{mort}_{t+1}\right)\mu_t + \tau^W_{t+1} + x_{t+1}\right\} p^D_t}{1+\overline{r}^B_{t+1}} - \frac{\left(1-\delta^d_{t+1}\right) p^D_{t+1} + \alpha^{Land}_{t+1} p^D_t}{1+\overline{r}^B_{t+1} + \psi^{Risk}_{t+1}}$$

2.7 Household Income

Recall from subsection 2.2 that the budget constraint of the household is

$$B_{a,t} = B_{a-1,t-1} + r_{a,t} B_{a-1,t-1} + y_{a,t} - f\left(D_{a,t}, D_{a-1,t-1}\right) - p^c_t C_{a,t}$$

Wealth B denotes non-housing net financial assets and excludes pension wealth. It includes ownership of financial stocks and bonds, as well as bank deposits, and subtracts non-mortgage bank debt.

The income term $y_{a,t}$ in this budget constraint incorporates a large number of taxes and transfers as well as the exogenous expenditure in rental housing. Before we detail the different elements inside $y_{a,t}$ it is useful to briefly define the rest of the items in the budget constraint.

The object f contains all items of the budget constraint that relate to owned housing and consists of total net expenditure on owned housing. The term $p^c_t C_{a,t}$ denotes all non-housing consumption expenditure. These consumption prices include taxes. The rate of return on wealth $r_{a,t}$ is a portfolio rate of return.

2.7.1 Income

The income variable $y_{a,t}$ contains the following elements: wage income[8], $y^W_{a,t}$, net pension income, $y^{PY}_{a,t} - y^{PC}_{a,t}$, expenditure on rental housing, an assortment of income transfers, $y^G_{a,t}$, various taxes not related to housing or pensions, $T^\tau_{a,t}$, received bequests, $y^{Beq}_{a,t}$, net income flows associated with children, $y^{children}_{a,t}$, and residual items.

$$
\begin{aligned}
y_{a,t} &= y^W_{a,t} + y^{PY}_{a,t} - y^{PC}_{a,t} - R^{rent}_t H_{a,t} + y^G_{a,t} \\
&\quad - T^{Hh}_{a,t} + y^{Beq}_{a,t} + y^{children}_{a,t} + y^{other}_{a,t}
\end{aligned}
$$

Of all these different items, only labor market income is endogenous to the household as it depends on search decisions.

The tax object $T^\tau_{a,t}$ captures a large number of specific taxes. Income taxes, local taxes, property taxes, taxes on financial income from stocks, taxes on income from individually held companies, estate taxes, labor market specific taxes, etc. These taxes are grouped differently depending on the purpose. For example, property value taxes are removed and included in the housing term $f(\cdot)$.

2.8 Pensions

Pension income enters the disposable income of households as an exogenous income, and pension wealth satisfies accumulation consistency requirements which are also exogenous to the household.

MAKRO uses a simplified version of the detailed pension model in DREAM. The payment data is taken from the pension model from the Ministry of Finance (which builds on DREAM[9]) and aggregates into three pension types indexed $i \in \{'Alder', 'Kap', 'PensX'\}$: 1) pensions that have already been taxed (primarily *alderspension*, $'Alder'$), 2) capital pension (*kapitalpension*, $'Kap'$) taxed with a flat rate, and 3) the aggregate of other pensions, taxed when received by households (*ratepensioner*, *livrentepensioner* and *ATP*, indexed $'PensX'$).

[8] Wage income per person is given by $w_t \rho_{a,t} h_{a,t} q^e_{a,t}$ where w_t is the wage per productivity unit, $\rho_{a,t}$ is productivity units per hour worked, $h_{a,t}$ is hours worked per full time worker and $q^e_{a,t}$ are full time workers per person. All j household types earn the same income given age. More detail can be found in the labor market chapter.

[9] In DREAM pensions are gender specific. As each generation in MAKRO is a cohort average, we sum pension contributions paid by men and women into their pension funds, as well as pensions received by men and women.

2.8.1 Pension Contributions

It is assumed that an exogenous part of wages is paid as contribution to each type (i) of pension:

$$y^{PC}_{i,a,t} = \lambda^{PC}_{i,a,t} y^{W}_{a,t}$$

where $y^W_{a,t}$ is wage income, and the parameter $\lambda^{PC}_{i,a,t}$ is calibrated so the contribution to pension type i matches the pension data from the Ministry of Finance.[10]

2.8.2 Pension wealth

The pension fund is a zero-profit vehicle with no costs. The three different types of pensions, indexed by i, are modeled as three separate actuarially fair pension schemes. So all contributions and after tax returns must be paid out to the same cohort (or as inheritance to their relatives).

The law of motion for pension wealth $B^P_{i,a,t}$ in a given pension scheme i is

$$B^P_{i,a,t} = \left\{ \left(1 + r^P_t \left[1 - \tau^P_t\right]\right) B^{P,i}_{a-1,t-1} \frac{N_{a-1,t-1}}{N_{a,t}} \right\} \left\{1 - \alpha^{BeqP}_{i,a,t} \left(1 - s_{a-1,t-1}\right)\right\} + y^{PC}_{i,a,t} - y^{PY}_{i,a,t}$$

The first term, $\left(1 + r^P_t \left[1 - \tau^P_t\right]\right) B^{P,i}_{a-1,t-1} \frac{N_{a-1,t-1}}{N_{a,t}}$, is the after-tax return on the wealth stock scaled with population as only the surviving current population shares the stock of wealth[11]. The second term, $1 - \alpha^{BeqP}_{i,a,t} \left(1 - s_{a-1,t-1}\right)$, adjusts for bequests. The fraction $1 - s_{a-1,t-1}$ does not survive from last period and a share of their pension wealth, $\alpha^{BeqP}_{i,a,t}$, is paid out to relatives as bequests and does not stay in the pension fund.

2.8.3 Pension Income

We assume that an exogenous age specific share of the start-of-period pension wealth, $B^P_{i,a-1,t-1}$, (plus returns and contributions)[12] is paid out and received by households each period as pension income $y^{PY}_{i,a,t}$:

$$y^{PY}_{i,a,t} = \lambda^{PY}_{i,a,t} \left[\left\{ \left(1 + r^P_t \left[1 - \tau^P_t\right]\right) B^{P}_{i,a-1,t-1} \frac{N_{a-1,t-1}}{N_{a,t}} \right\} \left\{1 - \alpha^{BeqP}_{i,a,t} \left(1 - s_{a-1,t-1}\right)\right\} + y^{PC}_{i,a,t} \right]$$

The parameter $\lambda^{PY}_{i,a,t}$ is calibrated to match pension income received by households in data from the Ministry of Finance.

10 $\lambda^{PC}_{\text{Kap},a,t}$ is the average share of wage income paid to capital pensions.
11 Note that r^P_t is the return net of financial costs.
12 We define the payout share as a share of start-of-period wealth plus any returns and contributions, such that setting the payout rate to one means ending the period with zero wealth in the particular pension scheme.

2.8.4 Composition and returns of pension portfolio

The aggregate pension wealth of pension fund i, $B^P_{i,t}$, is invested in stocks and bonds. We assume that all pension schemes have the same portfolio and yield the same return.

Assets of a specific type, k, held by the pension fund, $A^P_{k,t}$, are an exogenous fraction of total pension wealth:[13]

$$A^P_{k,t} = \omega^P_{k,t} \cdot B^P_t$$

The return r^P_t is a weighted average of the return on the portfolio elements:

$$r^P_t = \sum_k r_{k,t} \cdot \omega^P_{k,t}$$

2.9 Bequests

Warm glow utility from bequests allows the model to replicate the large amounts of wealth held at the late ages of the life cycle.

2.9.1 Death

Death occurs before any decisions are taken. One way to think of the timing is as follows. At the beginning of period t agents wake up or have died. In either case, the agent receives the financial income associated with their start-of-period net wealth. The assets of dead agents are distributed amongst their heirs as an exogenous income transfer. Finally, agents that are still alive receive other income and consume and save in period t, including out of bequests received.

2.9.2 Bequests received, liquidating housing, and bequests in utility

All agents leave and receive bequests. In the event of death houses are sold and mortgages are liquidated so that bequests received will consist of liquid assets plus the liquid value of the equity on the house after liquidation. Given the exogenous mortgage ratio relationship the equity that is transformed into liquid assets next period is

$$(1 - \mu_{a,t})\, p^D_{t+1} D_{a,t}$$

where $\mu_{a,t}$ is the mortgage ratio taken from the data and detailed below. This is then taxed and the resulting net value received by the multiple heirs. From an accounting perspective, bequests given and received must add up to the same amount, before taxes. The mapping from bequests given to bequests received is done with an allocation age$\times$age matrix M_t estimated using administrative data similarly to Boserup, Kopczuk, and Kreiner (2016) and detailed in Appendix 2.13.1.

[13] Assets are indexed by $k \in \{$bonds, domestic equities, foreign equities$\}$ and $B^P_t = \sum_i B^P_{i,t}$.

Bequests affect not only the budget constraint, but also preferences via $V_{a,t}^{Beq}$, which will be further explained in the section detailing all preference features.

2.10 Household financial portfolio

The budget constraint of a household is

$$B_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} + y_{a,t} - p_t^c C_{a,t} - f(D_{a,t}, D_{a-1,t-1})$$

The model endogenously determines the net financial variable B. In the data this quantity is made up of the sum of different assets $(\text{domestic equities, cash and bank deposits, foreign equities, bonds})$, indexed below by i, minus the sum of liabilities (bank debt), for generality indexed below by l, so that $B = \sum_i A_i - \sum_l L_l$.

We model this decomposition into assets and liabilities as functions of net non-mortgage financial assets B and housing D as well as an age-dependent intercept term I:

$$A_{i,a,t} = I_{i,a} Y_t^B + \lambda_i^B B_{a,t} + \lambda_i^D \bar{p}_t^D D_{a,t}$$

$$L_{l,a,t} = I_{l,a} Y_t^B + \lambda_l^B B_{a,t} + \lambda_l^D \bar{p}_t^D D_{a,t}$$

To preserve homogeneity, the intercept term is scaled with a slow-moving measure of average household income Y_t^B. Housing enters with the same gradually adjusting housing price, $\bar{p}_t^D$, that is used for the purpose of the mortgage loan-to-value fraction.

The marginal changes in the portfolio from a change in either B or D are independent of age and household type. All households therefore face the same marginal rates of return on net savings:

$$\begin{aligned} 1 + \bar{r}_{t+1}^B &= \frac{\partial}{\partial B_{a,t}} \{(1 + r_{t+1}) B_{a,t}\} \\ &= 1 + \sum_i r_{i,t+1} \lambda_i^B - \sum_l r_{l,t+1} \lambda_l^B \end{aligned}$$

and additional housing affects the portfolio in the same way:

$$\begin{aligned} \bar{r}_{t+1}^D &= \frac{\partial}{\partial D_{a,t}} \{(1 + r_{t+1}) B_{a,t}\} \\ &= \bar{p}_t^D \left(\sum_i r_{i,t+1} \lambda_i^D - \sum_l r_{l,t+1} \lambda_l^D \right) \end{aligned}$$

Appendix 2.13.2 covers the estimation of the household portfolio model and goes into more detail about the motivation behind the modeling decisions. There may be theoretical reasons to think portfolio composition should vary over the life cycle. If the data contains such heterogeneity, the estimated parameters will reflect that by having different values for different assets. In addition, the estimated portfolio is an optimal portfolio given the underlying assumption is that agents made optimal decisions that resulted in the portfolios we observe in the data.

2.11 Utility

2.11.1 Utility of consumption and housing

The main utility flow U is a CRRA object over a CES consumption composite $\tilde{U}$

$$U\left(C_{a,t}, D_{a,t}\right) = \frac{1}{1-\eta}\left[\tilde{U}_{a,t}\right]^{1-\eta}$$

The function $\tilde{U}_{a,t}$ is

$$\tilde{U}_{a,t} \equiv \left[\left(\upsilon^c_{a,t}\right)^{\frac{1}{E}}\left(\tilde{C}_{a,t}\right)^{\frac{E-1}{E}} + \left(\upsilon^d_{a,t}\right)^{\frac{1}{E}}\left(\tilde{D}_{a,t}\right)^{\frac{E-1}{E}}\right]^{\frac{E}{E-1}}$$

where E is the substitution elasticity between housing and non-housing, and υ^c, υ^d are utility weights on the two. The contemporaneous derivatives are

$$U^c_{a,t} \equiv \frac{\partial U_{a,t}}{\partial \tilde{C}_{a,t}} = \left[\tilde{U}_{a,t}\right]^{-\eta}\left(\frac{\upsilon^c_{a,t}\tilde{U}_{a,t}}{\tilde{C}_{a,t}}\right)^{\frac{1}{E}}$$

and

$$U^d_{a,t} \equiv \frac{\partial U_{a,t}}{\partial \tilde{D}_{a,t}} = \left[\tilde{U}_{a,t}\right]^{-\eta}\left(\frac{\upsilon^d_{a,t}\tilde{U}_{a,t}}{\tilde{D}_{a,t}}\right)^{\frac{1}{E}}$$

Finally, non-housing consumption is itself a CES nest over a range of consumption goods, see subsection 2.12.

2.11.2 Habit formation and reference consumption

Recent evidence on inter-temporal consumption behavior shows that following a one-time transfer, consumption rises for several periods with a mean on-impact yearly marginal propensity to consume (MPC) of roughly 50 percent, see Fagereng et. al. 2021. This is inconsistent with the typical business cycle models; the standard way to match high on-impact MPC's has been to combine a share of Ricardian agents with a share of "hand-to-mouth" (HtM) agents who consume all income in every period (e.g. Bilbiie 2008). This, however, only generates high on-impact marginal propensities to consume but no effects in the following periods (as the Ricardian agents have close to zero MPC's for transitory shocks), and is thus contrary to empirical evidence showing significant year $t+1$ effects. Recent business cycle literature has solved this by introducing microeconomic stochasticity and incomplete markets (Heterogeneous-Agent New-Keynesian, HANK, models), in which agents close to but not at their borrowing constraint will have high MPC's and smooth consumption.[14] Here, we implement a solution that does not rely on stochasticity.[15] In particular, we combine a representative agent model with consumption habits, as is typical in the Dynamic Stochastic General Equilibrium (DSGE) literature, with a share of reference consumption that tracks (liquid) income, and therefore generates a high on-impact MPC while also generating persistence (year $t+1$ effects) due to habits. The idea of including income as a term in reference consumption was first presented in Carroll et al. (2023). We refer to this component as HtM consumption. The resulting high MPC behavior can be interpreted in various ways. For example, high on-impact spending of a transitory income shock is typically explained by liquidity constraints in the two-agent New-Keynesian literature (TANK), e.g., models with hand-to-mouth consumers and the HANK literature. A different contender for high first-period excess spending that does not rely on liquidity constraints is durable (non-housing) purchases. Durables have layouts that are more front-loaded due to the long-lasting consumption flow of the stock cf. Laibson et al. (2022).

We re-write utility as

$$\tilde{U}_{a,t} \equiv \left[\left(v^c_{a,t}\right)^{\frac{1}{E}} \left(\tilde{C}_{a,t}\right)^{\frac{E-1}{E}} + \left(v^d_{a,t}\right)^{\frac{1}{E}} \left(\tilde{D}_{a,t}\right)^{\frac{E-1}{E}} \right]^{\frac{E}{E-1}}$$

where $\tilde{C}_{a,t}$ denotes consumption net of a reference quantity:

$$\tilde{C}_{a,t} = \frac{C_{a,t}}{\zeta_{a,t}} - \chi^C \frac{C_{a-1,t-1}}{\zeta_{a-1,t-1}} - \gamma \frac{y_{\text{HtM},a,t}}{P^C_t}$$

where χ^Cdenotes the (external) reference share of consumption and γ is the reference share on income with $y_{\text{HtM},a,t}$ being a liquid income term defined as

$$\begin{aligned} y_{\text{KC},a,t} = y_{a,t} & \\ & - \text{capital pensions} - \text{age pension} \\ & - \text{inheritance adjustment} \end{aligned} \tag{2.6}$$

We assume that households do not have high marginal propensities to consume out of age pension, capital pension and an inheritance adjustment term[16]

[14] In order to get enough initial period impact the literature contains different approaches such as two assets (Kaplan et al., 2018), discount factor heterogeneity (e.g. Hagedorn et al. 2019), or a generalized habit (Carroll et al. (2023)).

[15] Note that it is possible to solve a particular class of HANK models without numerical stochasticity by assuming zero-liquidity (zero savings), see e.g. Bilbiie 2020. This is not possible in our case, since we want to match the model to the rich features of wealth in data. Also note that a recent strand of literature obtains empirically realistic inter-temporal MPC's, see e.g. Angeletos et al. (2024), by building on a perpetual youth OLG model alla Blanchard (1985) with very high probabilities of death.

[16] These pensions are regularly payed out in large lump sums, which would imply unrealistic jumps in consumption. The inheritance adjustment term is a term to make up for the fact that when households die in a cohort, the dying households are typically poorer than the average household. Such re-distributive transfers would also induce

The weight, $\zeta_{a,t}$, depends on the number of children in the household

$$\zeta_{a,t} = 1 + \frac{1}{2} n_{a,t}^{\text{children}}$$

The reference quantity $C_{a-1,t-1}$ is viewed as the average of the cohort in the previous period, rather than the individual household's own previous decisions. In this way it is exogenous to the household.

We do the same for housing by considering the following object inside utility

$$\tilde{D}_{a,t} = \frac{D_{a,t}}{\zeta_{a,t}} - \chi^D \frac{D_{a-1,t-1}}{\zeta_{a-1,t-1}} - \gamma \frac{y_{\text{HtM},a,t}}{P_t^C}$$

Note that $\zeta_{a,t}$ can be eliminated by redefining utility weights $\upsilon_{j,a,t}^c$ and $\upsilon_{j,a,t}^d$ to include the effect of children implicitly (and slightly modifying the coefficients χ^C and χ^D). We include the explicit adjustment for the number of children as this number is also forecast explicitly, whereas $\upsilon_{a,t}^d$ and $\upsilon_{a,t}^c = 1 - \upsilon_{a,t}^d$ are calibrated to match the level of consumption in the data.

Housing habits are internal so the household takes them into account in optimization. See Dennis (2009) for a comparison of *internal* and *external* habits.

The contemporaneous derivatives with habits and household size are

$$U_{a,t}^c = \frac{\partial U_{a,t}}{\partial \tilde{C}_{a,t}} \frac{\partial \tilde{C}_{a,t}}{\partial C_{a,t}} = \left[\tilde{U}_{a,t}\right]^{-\eta} \left(\frac{\upsilon_{a,t}^c \tilde{U}_{a,t}}{\tilde{C}_{a,t}}\right)^{\frac{1}{E}} \frac{1}{\zeta_{a,t}}$$

and

$$U_{a,t}^d = \frac{\partial U_{a,t}}{\partial \tilde{D}_{a,t}} \frac{\partial \tilde{D}_{a,t}}{\partial D_{a,t}} = \left[\tilde{U}_{a,t}\right]^{-\eta} \left(\frac{\upsilon_{a,t}^d \tilde{U}_{a,t}}{\tilde{D}_{a,t}}\right)^{\frac{1}{E}} \frac{1}{\zeta_{a,t}}$$

2.11.3 Utility of Bequests

Bequests do not just enter the budget constraint. They are a key object in preferences. The bequest utility of the dying agent is

$$V_{a,t}^{Beq} = \xi_{a,t}^{Beq} \frac{\left[X_{a,t}^{Beq}\right]^{1-\eta}}{1-\eta} \tag{2.7}$$

where we define the interior object, X^{Beq}, as

$$X_{a,t}^{Beq} \equiv \left(1 - \tau_{t+1}^{Beq}\right) \frac{B_{a,t} + p_t^D \left(1 - \mu_{a,t}\right) D_{a,t} + V_{a,t}^{PensionB}}{p_{t+1}^c} - \xi^0$$

and the utility associated with bequests is scaled with $\xi_{a,t}^{Beq}$.[17] It is important to note that, as this is a utility construction, there is a degree of freedom in the definition of the object X^{Beq}.

unrealistic jumps in consumption if included in the Keynesian consumption income term.

[17] In these functions we generally allow for an interior parameter ξ^0 to avoid negative objects rendering the model unsolvable. This parameter has a default value of zero.

Here we attach value to the sum of assets, rather than, for example, attaching a separate special value to the house, although both formulations are feasible.[18] Using the sum of assets inside X^{Beq} implies all assets have the same marginal utility of bequest motive. Finally, note that we value the house inside X^{Beq} at the current price p_t^D. The derivatives of (2.7) are

$$\frac{\partial V_{a,t}^{Beq}}{\partial B_{a,t}} = \frac{\partial V_{a,t}^{Beq}}{\partial X_{a,t}^{Beq}} \frac{\partial X_{a,t}^{Beq}}{\partial B_{a,t}} = \xi_{a,t}^{Beq} \left[X_{a,t}^{Beq}\right]^{-\eta} \frac{\left(1-\tau_{t+1}^{Beq}\right)}{p_{t+1}^c}$$

and

$$\frac{\partial V_{a,t}^{Beq}}{\partial D_{a,t}} = \frac{\partial V_{a,t}^{Beq}}{\partial X_{a,t}^{Beq}} \frac{\partial X_{a,t}^{Beq}}{\partial D_{a,t}} = \frac{\partial V_{a,t}^{Beq}}{\partial B_{a,t}} \underbrace{p_t^D \left(1-\mu_{a,t}\right)}_{\frac{\partial f_{a,t}}{\partial D_{a,t}}}$$

2.11.4 Utility of Wealth

Similarly to the bequest function, the utility of wealth is

$$V_{a,t}^{Wealth} = \xi_{a,t}^{Wealth} \frac{\left[X_{a,t}^{Wealth}\right]^{1-\eta}}{1-\eta} \tag{2.8}$$

where the elasticity η is the same as in the utility and bequest functions. We define the interior object, X^{Wealth}, as

$$X_{a,t}^{Wealth} \equiv \frac{B_{a,t} + p_t^D \left(1-\mu_{a,t}\right) D_{a,t} + V_{a,t}^{PensionW}}{p_{t+1}^c} - \xi^0$$

with the difference relative to bequests being the absence of the inheritance tax and the composition of the pension entitlement.

Note that we do not account for transaction costs incurred in liquidating or trading houses (or other assets) in our utility objects, $X_{a,t}^{Wealth}$ or $X_{a,t}^{Beq}$.

The pension entitlements inside bequest utility and wealth utility differ. Regarding bequest utility, some pension plans cease at death. In contrast, others contain an insurance element and continue to pay descendants. $V_{a,t}^{PensionB}$ is the amount from pensions paid to descendants in case of death. Regarding utility from wealth, once households reach a certain age, they can freely choose the payout schedule from 'Kapitalpension' and 'aldersopsparing'. Many choose to withdraw at the latest possible date, 20 years after retirement, when the whole amount is paid out at once. Therefore the object $V_{a,t}^{PensionW}$ differs from the pension object inside bequest utility.

2.11.5 Disutility of house price changes.

To better account for house price momentum (positive autocorrelation in price changes), we introduce a disutility of house price changes that captures various factors affecting buyers' and

[18] Li, Liu, Yang, and Yao (2016) have a CES function of housing and other assets as bequest utility. Kaplan, Mitman and Violante (2017) do as here.

sellers' decisions. This price momentum can be micro-founded in various ways following the literature on house price sensitivity. For example, when selling a house, reference price utility, as described in Barberis and Xiong (2012), leads households to experience disutility from selling below the purchase price. Sellers may also be loss-averse in reference pricing, incentivizing them to hold house prices higher for longer than optimal given actual demand, in order to avoid the marginal loss discontinuity at the reference point. Additionally, demand concavity, as observed in Guren (2018), means that if prices fall below a certain threshold, demand does not increase (since households typically only need one house). Finally, downsizing aversion, due to financial constraints such as the need for a down-payment on a new house, as described in Stein (1995), may further discourage sellers from lowering their prices. These effects have been shown to be important in Danish data on the seller side, as documented in Andersen et al. (2022). On the other hand, for buyers, recent sale prices and expected profits may serve as reference points for making purchasing decisions. Bao and Saunders (2021) found, based on micro-data from the UK housing market, that buyers use these reference points to guide their purchase behavior. We aim to capture these various incentives in the following way;

We temporarily shift the preference for housing, $\upsilon^d_{a,t}$, when prices are volatile through the dampening function Ψ^D_t

$$\upsilon^d_{a,t} = \hat{\upsilon}^d_{a,t} \cdot \left(1 - \Psi^D_t\right)$$

with the functional form

$$\Psi^D_t = \Xi\left(\Pi^D_t, \Pi^D_{t-1}\right) - \beta_j 2\Xi\left(\Pi^D_{t+1}, \Pi^D_t\right) \tag{2.9}$$

where

$$\Xi\left(\Pi^D_t, \Pi^D_{t-1}\right) = \mu_{pd}\left(\left[\frac{\Pi^D_t}{\Pi^D_{t-1}} - 1\right]\frac{\Pi^D_t}{\Pi^D_{t-1}}\right) \tag{2.10}$$

and $\Pi^D_t \equiv P^D_t / P^D_{t-1}$ is gross house price inflation. The parameter μ_{pd} is a scaling constant.

2.11.6 Differentiating with respect to housing

In the housing first order condition we have the object

$$U^d_{a,t} + \beta s_{a,t} \frac{\partial U_{a+1,t+1}}{\partial D_{a,t}}$$

and it is useful to make this expression explicit here. The first element is the contemporaneous derivative

$$\begin{aligned} U^d_{a,t} &= \left[\tilde{U}_{a,t}\right]^{-\eta} \frac{\partial \tilde{U}_{a,t}}{\partial \tilde{D}_{a,t}} \frac{\partial \tilde{D}_{a,t}}{\partial D_{a,t}} \\ &= \left[\frac{\upsilon^d_{a,t}}{\tilde{D}_{a,t}}\right]^{\frac{1}{E}} \left[\tilde{U}_{a,t}\right]^{\frac{1}{E}-\eta} \frac{1}{\zeta_{a,t}} \end{aligned}$$

while the second is the derivative in period t+1 with respect to current housing

$$\frac{\partial U_{a+1,t+1}}{\partial D_{a,t}} = \left[\tilde{U}_{a+1,t+1}\right]^{-\eta} \cdot \frac{\partial \tilde{U}_{a+1,t+1}}{\partial \tilde{D}_{a+1,t+1}} \frac{\partial \tilde{D}_{a+1,t+1}}{\partial D_{a,t}}$$
$$= -\frac{\chi^D}{\zeta_{a,t}} \left[\frac{\upsilon^d_{a+1,t+1}}{\tilde{D}_{a+1,t+1}}\right]^{\frac{1}{E}} \left[\tilde{U}_{a+1,t+1}\right]^{\frac{1}{E}-\eta}$$

2.12 Consumption components

At the top of the household utility function we have two components: owned housing and the non-housing consumption aggregate. Owned housing is a single good with no sub-components but non-housing consumption aggregates many elements through a CES tree structure. Note, however, that rental housing is not an element in the CES tree, but instead it is an exogenous element in the budget constraint of the household.

The optimal choice of total consumption, savings, and housing, is described above in section 2.2. In this section we detail the determination of the components of total non-housing consumption, $C_{a,t}$. The first decomposition of this object contains five different components.

$$c \in \{\text{cars, energy, goods, tourism, services}\}$$

Household demand for these five consumption components is a part of total demand for output from the nine domestic sectors as well as for imported goods, a process described in the input/output chapter.

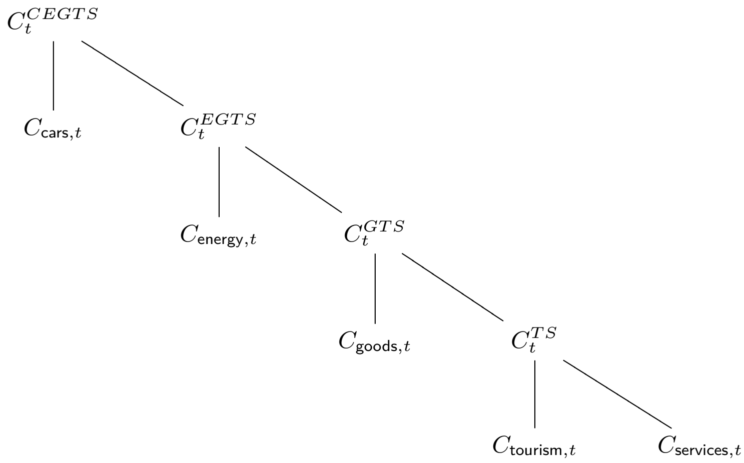
Note that services included in the non-housing consumption aggregate does not include expenses to financial services (from the pension sector). Expenses to financial services are for example expenses to administration and investment of pension funds. For the consumers these are costs that reduce their financial returns on these assets. In the national accounts these costs are used to produce financial services for the households - i.e. consumption of services. The aggregate consumption of services include these financial services as does aggregate consumption, but they are not part of the nested consumption excluding housing. It is assumed that financial services has the same price as other services.

2.12.1 The nested consumption components

These five consumption components come together in the following CES nest structure, where the uppermost aggregate for an agent aged a at time t is $C_{a,t}$. To keep the model tractable and due to data limitations, we assume that all agents have identical preferences for these goods. We can therefore aggregate individual agent non-housing consumption as

$$C_t^{CEGTS} = \sum_a C_{a,t}$$

We can now detail the CES nest structure using only aggregate consumption:



2.12.2 CES optimization

The approach of nested CES cost minimization is described in detail in the production chapter. The problem here is identical, only simpler as there are no extra elements such as technological progress or variable utilization multiplying consumption quantities. We can summarize the problem at every level of the consumption tree as follows

$$\text{Utility} \quad \Rightarrow \quad C^{ij} = \left[\left(\mu^i\right)^{\frac{1}{\eta}} \left(C^i\right)^{\frac{\eta-1}{\eta}} + \left(\mu^j\right)^{\frac{1}{\eta}} \left(C^j\right)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$\text{Derivative} \quad \Rightarrow \quad \frac{\partial C^{ij}}{\partial C^i} = \left(\mu^i \frac{C^{ij}}{C^i}\right)^{\frac{1}{\eta}}$$

$$\text{Demand/F.O.C.} \quad \Rightarrow \quad C^i = \mu^i C^{ij} \left(\frac{P^{ij}}{p^i}\right)^{\eta}$$

$$\text{Constraint} \quad \Rightarrow \quad P^{ij} C^{ij} = p^i C^i + p^j C^j$$

$$\text{CES Price} \quad \Rightarrow \quad P^{ij} = \left[\mu^i \left(p^i\right)^{1-\eta} + \mu^j \left(p^j\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

Here the μ are scale parameters, and the η are elasticities. Table 2.1 shows the estimated elasticities in each of these nests (Kronborg & Kastrup, 2020).

Table 2.1
Upper Tree Elasticities (η)

	η
C and Housing	1
Cars and C^{EGTS}	1.04
Energy and C^{GTS}	0.26
Goods and C^{TS}	0.94
Tourism and services	1.25

2.12.3 Tourism

There are both imports and exports of tourism. Imports of tourism consist of how much Danish households consume abroad and are given by the demand component $C_{\text{tourism},t}$ from the tree above. This is a regular consumption good and its demand increases with income. Exports of tourism are determined in the exports chapter and its aggregate is $X_{\text{tourism},t}$. The consumption of foreign tourists in Denmark is also divided into consumption groups in the export chapter and is given by $C_{c,t}^{Tourist}$.

Total consumption of goods type c in Denmark is the sum of Danish households' consumption and tourists' consumption in Denmark:

$$P_{c,t}^{CDK} C_{c,t}^{DK} = P_{c,t}^{C} C_{c,t} + P_{c,t}^{CTourist} C_{c,t}^{Tourist}$$

The input/output system is based on deliveries from the 9 production sectors to demand groups (see input/ouput chapter). The demand groups for consumption include tourists' consumption, $C_{c,t}^{DK}$.

Whereas in the model Danes and tourists face the same prices for the same goods, in order to match the data they cannot face the same price for the same consumption components. We therefore use an adjustment factor

$$P_{c,t}^{CTourist} = \lambda_{c,t}^{pCTourist} P_{c,t}^{C}$$

where $\lambda_{c,t}^{pCTourist}$ is a parameter used to fit the data. It is assumed that this price margin remains constant going forward.

The value of aggregate Danish consumption does not include the consumption of foreign tourists in Denmark (as these are counted as exports).

2.13 Appendices

In the following appendices the emphasis is on the derivation of parts of the model.

2.13.1 Calculating the bequest allocation matrix

This section follows Boserup, Kopczuk, and Kreiner (2016). Households take bequests received as exogenous and these enter the budget constraint as an additive term which is "hidden" inside the income variable. Even if agents receive bequests in the first period of economic life (age 18), we still have the initial condition for assets that $B_{0,t}$ is taken as given by the agent, as we exogenize transfers associated with children.

An individual of any given age receives bequests from agents deceased also at any given age. The distribution of bequests is modeled through a time varying matrix $M_t(a_d, a_h)$ where the indices refer respectively to the age of the deceased and to the age of the heir. This allocation matrix is general in that it encompasses all deaths, not just deaths of parents or grandparents, and all heirs.[19] As an example, children also die and leave assets to their parents and siblings.

When an individual dies in the model, he leaves a bequest. Assume that the individual dies at age a_d. The distribution matrix $M_t(a_d, a_h)$, describes the share of his bequest going to an average a_h year old individual. A given fraction of his wealth which he leaves as bequest is distributed equally by all agents of age a_h.

The distribution matrix. The matrix $M_t(a_d, a_h)$ is based on estimates of individual bequests from Danish administrative data. These estimates are obtained using a difference-in-difference estimator. This measures how the difference in wealth of an individual of age a_h, whose relative of age a_d has died, differs from the the difference in wealth of the average person of age a_h. This results in estimates of several specific bequests from a_d year old individuals to a_h year old individuals. Let i be the index for each specific transfer from an a_d year old to an a_h year old. $\widetilde{H}_{a_d,a_h,i,t}$ is then the estimated nominal amount transferred for each specific transfer. These bequests given by individuals in age group a_d to individuals in age group a_h are then summed and divided by the total number of a_d and a_h year olds (not just the ones involved in estimated bequest transfers but all individuals). The result is a data frame containing the average bequest $H_{a_d,a_h,t}$ received by an a_h year old from an a_d year old, regardless of whether a relative has died,

$$H_{a_d,a_h,t} = \frac{1}{N_{a_h,t}N_{a_d,t}} \left[\underbrace{\sum_i \widetilde{H}_{a_d,a_h,i,t}}_{\text{All transfers } a_d \text{to } a_h} \right]$$

19 The sample is larger than in Boserup, Kopczuk, and Kreiner, (2016). Nearly everyone who dies has a son or daughter, a parent, a nephew or niece, an uncle, etc. Therefore, unaccounted would be only those who die alone yet have substantial assets to distribute.

where $N_{x,t}$ is the number of people of age group x. The age groups range from 0 to 100 and the time span is from 2000 to 2012.[20] The average bequest given/left by an individual from age group a_d is then given by

$$H_{a_d,t} = \sum_{a_h} H_{a_d,a_h,t} N_{a_h,t} = \frac{1}{N_{a_d,t}} \left[\underbrace{\sum_{a_h} \left(\sum_i \widetilde{H}_{a_d,a_h,i,t} \right)}_{\text{All transfers } a_d \text{to all } a_h} \right]$$

Due to the sparsity of these matrices, they are averaged over time

$$H_{a_d} = \frac{1}{T} \sum_t^T H_{a_d,t}, \qquad H_{a_d,a_h} = \frac{1}{T} \sum_t^T H_{a_d,a_h,t}$$

The share of an a_d year old's bequests received by an a_h year old is then

$$\widetilde{\chi}_{a_d,a_h} = \frac{H_{a_d,a_h}}{H_{a_d}}$$

This share contains a large amount of noise. We therefore conduct a non-parametric estimation, using a local linear regression with the age of both giver and receiver as dependent variables and a Gaussian kernel. $\widetilde{\chi}_{a_d,a_h}$ is then replaced by the fitted value χ_{a_d,a_h}.
Since χ_{a_d,a_h} is time invariant, $\chi_{a_d,a_h} N_{a_h,t}$ will generally not sum to 1. This means that bequests given will not be the same as bequests received. To prevent this, the shares are normalized so that we finally obtain the allocation matrix

$$M_t(a_d, a_h) = \frac{\chi_{a_d,a_h}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}}$$

These have the desired property that

$$\sum_{a_h} M_t(a_d, a_h) N_{a_h,t} = \frac{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}} = 1$$

Therefore total bequests given will equal total bequests received.

Consistency. Since people die at the end of a period the total bequest given by a deceased member of age group a_d, consists of his assets at the end of the period, which in the case of the model are net financial assets $B_{a_d,t}$ and housing.[21] Here we proceed using only net financial assets B as an illustration. This means that the average bequest given by a member of age group a_d is $(1 - s_{a_d,t}) B_{a_d,t}$ where $s_{a_d,t}$ is the survival rate, i.e. the probability of an a_d year old also being alive at age $a_d + 1$. Total bequests given by all age groups at the end of

[20] Note that age zero in the data corresponds to index 1 of age in the model.
[21] In the data the value of property is included in the wealth difference such that the allocation matrix we use is consistent with the model.

time t after all decisions have been taken are then

$$H_t = \sum_{a_d} (1 - s_{a_d,t}) B_{a_d,t} N_{a_d,t}$$

Bequests are received in the next period. The average bequest from an a_d year old deceased at the end of period $t-1$ received by an a_h year old in period t will therefore be

$$(1 - s_{a_d,t-1}) B_{a_d,t-1} M_t (a_d, a_h)$$

The bequest received by a member of age group a_h at time t is then given by

$$H_{a_h,t} = \sum_{a_d} (1 - s_{a_d,t-1}) B_{a_d,t-1} M_t (a_d, a_h) N_{a_d,t-1}$$

This in turn results in total bequests received being

$$\begin{aligned}
\sum_{a_h} H_{a_h,t} N_{a_h,t} &= \sum_{a_h} \left(\sum_{a_d} (1 - s_{a_d,t-1}) B_{a_d,t-1} M_t (a_d, a_h) N_{a_d,t-1} \right) N_{a_h,t} \\
&= \sum_{a_h} \left(\sum_{a_d} (1 - s_{a_d,t-1}) B_{a_d,t-1} \frac{\chi_{a_d,a_h}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}} N_{a_d,t-1} \right) N_{a_h,t} \\
&= \sum_{a_d} (1 - s_{a_d,t-1}) B_{a_d,t-1} \left(\sum_{a_h} \frac{\chi_{a_d,a_h} N_{a_h,t}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}} \right) N_{a_d,t-1} \\
&= \sum_{a_d} (1 - s_{a_d,t-1}) B_{a_d,t-1} N_{a_d,t-1} = H_{t-1}
\end{aligned}$$

so that total bequests given last period equal total bequests received this period[22].

2.13.2 Estimating assets and Liabilities as functions of B

The model generates the net financial variable B endogenously. In the data this quantity is made up of the sum of different assets (stocks, bonds and bank deposits) minus the sum of liabilities (bank debt), so that in gross term $B = A - L$. This decomposition of B into assets and liabilities displays systematic patterns over the life cycle, and here we detail how to capture these features and use them in our model. We first look at a very simple example to highlight the approach.

Assets and Liabilities as functions of B The exogenous portfolio composition is estimated from the data as in the following example with one asset and one liability. This allows us to take into account that 1) portfolios change with age and 2) portfolios change with the size of net-savings. Assets $A_{a,t}$ are related to net financial wealth $B_{a,t}$ through the equation

$$A_{a,t} = I_A + \lambda B_{a,t}$$

[22] The amount actually available as disposable income for the receiver differs from the amount given due to transaction costs, taxes and interests payments.

and we have the same for liabilities

$$L_{a,t} = I_L + \phi B_{a,t}$$

and here the quantities $A_{a,t}$ and $B_{a,t}$ are cohort averages, and therefore the quantities that apply to individual households.

Since $A_{a,t} - L_{a,t} = B_{a,t}$ we must have that our estimated parameters obey $I_L = I_A$ and $\lambda - \phi = 1$. As it turns out these properties are ensured by the mechanics of OLS estimation. The OLS regressions are $A = X\beta_A + \epsilon_A$, and $L = X\beta_L + \epsilon_L$ where A and L are column vectors. Taking the estimator for assets, $\hat{\beta}_A = (X'X)^{-1}X'A$, and using the fact that $A = B + L$, we can write $\hat{\beta}_A = (X'X)^{-1}X'B + \hat{\beta}_L$. Here, since the matrix X has two column vectors, a column of ones and the vector B, the OLS algebra implies

$$\hat{\beta}_A - \hat{\beta}_L = (X'X)^{-1}X'B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In a regression $A = X\beta + \epsilon$ the OLS estimator of β projected A on the space generated by the columns of X, containing the vector B. Since $A - L = B$ the sum across equations of coefficients attached to B must be 1 and the sum of coefficients attached to any other variables must be 0, because the projection of B on B is exact. These properties extend to the case where the regressions have explanatory variables so that the matrix X has more columns.

OLS is adequate as the relationships we are estimating are not behavioral. They are instead a way to capture the patterns observed in the data more accurately than just using averages as done in e.g. DREAM. For historical data we add the OLS error to the estimated regression so as to replicate the portfolio data exactly.[23] For forward looking simulation we leave the orthogonal error (which has mean zero across ages and across assets) out, and use the estimated parameters, plus endogenous B and any other explanatory variables inside X to generate a portfolio going forward. Note that the estimation procedure and the exact specification of these equations are agnostic (given the specification assumed) with respect to the data. There may be theoretical reasons to think portfolio composition should vary over the life cycle. If the data contains such heterogeneity, the estimated parameters will reflect that by having different values for different assets. In addition, the estimated portfolio is an optimal portfolio given the underlying assumption is that agents made optimal decisions that resulted in what we observe.

General Structure For several asset types i, and liability types j, we have at any given moment t:[24]

$$A^i_{a,t} = I^i_{a,t} + \lambda^i_t B_{a,t}$$
$$L^j_{a,t} = I^j_{a,t} + \phi^j_t B_{a,t}$$

where $I^i_{a,t}$ and $I^j_{a,t}$ are the intercept functions for each age and (λ^i_t, ϕ^j_t) are the parameters associated with net assets. The intercept terms $I_{a,t}$ are general functions of age and of auxiliary variables with population, aggregate net assets B, and another macroeconomic

[23] OLS is unbiased. A GLS estimator with weights given by population N_a is more efficient in the presence of age related heteroskedasticity. However, the error from this estimator, $Y - X\hat{\beta}_{GLS}$, no longer sums to zero.
[24] We only run regressions along the age dimension. When data allows we will look into panel estimation.

variable Q as a scaling factors. With a first order polynomial in age and with one auxiliary variable z the intercept is as follows

$$I^{i,j}_{a,t} = I^{i,j}_{0,t}\left[\frac{Q_t}{N_t}\right] + I^{i,j}_{1,t}\left[\frac{Q_t}{S^a_t}\right]a + I^{i,j}_{z,t}Z_{a,t}$$

with $N_t = \sum_a N_{a,t}$, and $S^a_t = \sum_a N_{a,t}\cdot a$, and $B_t = \sum_a N_{a,t}B_{a,t}$ and $Z_t = \sum_a N_{a,t}Z_{a,t}$. The object Q_t is a macro variable. We can also have several Z_t variables to fit the portfolio.

Homogeneity We now briefly drop the time subscript. In OLS regression terminology, the objects Y and X are (for asset i) given by $Y = \left[A^i_a\right]$ with X containing four column variables (intercept, age, Z, and B) with as many rows as ages a:

$$X = \left[\begin{array}{cccc}\frac{Q}{N}, & \frac{Q}{S^a}a, & Z_a, & B_a\end{array}\right]$$

The way the intercept terms are defined plays a role in ensuring homogeneity of degree 1 in the model. Homogeneity of degree 1 is ensured if, when increasing all exogenous variables by a common factor Λ, the model yields all endogenous variables factored by the same Λ such that no relative quantities change. Consider the equation for asset A^i and aggregate over all ages a and all households N_a . Inserting the intercept we have

$$\sum_a N_a A^i_a = \sum_a N_a\left\{I^i_0\left[\frac{Q}{N}\right] + I^i_1\left[\frac{Q}{S^a}\right]a + I^i_z Z_a\right\} + \lambda^i\sum_a N_a B_a$$

which, after canceling terms, yields

$$\frac{A^i}{B} = \left[\left(I^i_0 + I^i_1\right)\frac{Q}{B} + I^i_z\frac{Z}{B} + \lambda^i\right]$$

The portfolio structure only needs to ensure the homogeneity of aggregate (sum over all individuals and all cohorts) assets and liabilities with respect to B. The rest of the MAKRO model must then ensure that B is homogeneous with respect to all other variables. Since Q is a macro variable introduced as a scaling factor, the long-run portfolio composition is constant as long as MAKRO preserves the ratio Q/B. The same is true for the ratio Z/B. However, policy interventions or structural changes that affect the long-run ratios Q/B and Z/B will affect long-run portfolio compositions. Of particular note are the scaling objects N and S^a, which ensure that the portfolio is homogeneous with respect to population size.

Housing We include housing in the auxiliary variable Z in the regression since it is a key determinant of non-mortgage bank debt. Note that we implement the portfolio with one short-run deviation from the estimated model: a slow-moving house price instead of the market clearing price. We now detail the algebra associated with these features. Consider the equation for some asset i. It is estimated as

$$N_{a,t}A^i_{a,t} = N_{a,t}\left\{I^i_0\left[\frac{Q_t}{N_t}\right] + I^i_1\left[\frac{Q_t}{S^a_t}\right]a + I^i_d\cdot p^D_t D_{a,t} + \lambda^i B_{a,t}\right\}$$

and aggregation $\sum_a N_{a,t} A^i_{a,t} = A^i_t$ implies (where $p^D_t D_t$ is total housing)[25]

$$A^i_t = \left\{ I^i_0 + I^i_1 \right\} Q_t + I^i_d \left[p^D_t D_t \right] + \lambda^i B_t$$

$$A^i_{a,t} = I^i_0 \left[\frac{Q_t}{N_t} \right] + I^i_1 \left[\frac{Q_t}{S^a_t} \right] a + I^i_d \cdot \left[X_t \cdot \bar{p}^D_t D_{a,t} \right] + \lambda^i B_{a,t}$$

The house price $\bar{p}^D_t$ attached to $D_{a,t}$ in this equation is the same for all ages and therefore just scales the estimated parameter I^i_d. As we implement this equation we define this price as a moving average: $\bar{p}^D_t = \left(1 - \kappa^{PD}\right) p^D_t + \kappa^{PD} \bar{p}^D_{t-1}$ where p^D_t is the realized house price. In the long run $\bar{p}^D_t = p^D_t$.

The marginal return on housing is $I^i_d \cdot X_t \bar{p}^D_t$, and with all assets and liabilities

$$\bar{r}^D_t = \bar{p}^D_t \left(\sum_i r^i_t I^i_{d,t-1} - \sum_i r^j_t I^j_{d,t-1} \right)$$

Aggregation and long run properties, however, require some adjustment. As we aggregate $\sum_a N_{a,t} A^i_{j,a,t} = A^i_t$ the intercept and the age term become

$$\sum_a N_{a,t} \left\{ I^i_0 \left[\frac{Q_t}{N_t} \right] + I^i_1 \left[\frac{Q_t}{S^a_t} \right] a \right\} = \left\{ I^i_0 + I^i_1 \right\} Q_t$$

while the last term becomes simply B_t. Aggregating over population yields

$$I^i_d \sum_a N_{a,t} \bar{p}^D_t D_{j,a,t} = I^i_d \bar{p}^D_t D_t$$

which means the aggregate portfolio obeys

$$A^i_t = \left(I^i_0 + I^i_1 \right) Q_t + I^i_d \left[\bar{p}^D_t D_t \right] + \lambda^i B_t$$

so that homogeneity is a property determined by the ratio Q/B and the ratio $\frac{\bar{p}^D_t D_t}{B_t}$.

Note that, as the entire household problem generates endogenous variation for B and Z, the estimated portfolio model allows for endogenous variation of its constituent parts which by design is an optimal portfolio adjustment. Note also that any sluggishness in portfolio adjustment relative to economic conditions is already included in the estimated equations, either through lower coefficients attached to explanatory variables or through a higher intercept. Finally, the presence of the constant term yields an estimation error which is orthogonal to the life cycle, a property which is useful in the forecasting role of the model.

[25] The actual estimated equation multiplies both sides of this equation by population $N_{a,t}$. In addition, the scaling objects Q_t, N_t,and S^a_t are implicit as constants mutiplying the variables so that the actual coefficients (I^i_0, I^i_1) are recovered ex post by inversion. For example for the intercept we estimate $\hat{\beta}^i_0$ and then recover $\hat{I}^i_0 = (N_t/Q_t)\hat{\beta}^i_0$.

Marginal return on B Given a portfolio structure we now must fit the budget constraint on historical data. The budget constraint with explicit assets and liabilities is

$$B_{a,t} = B_{a-1,t-1} + \underbrace{\left[\sum_i r_t^i A^i\left(B_{a-1,t-1}\right) - \sum_j r_t^j L^j\left(B_{a-1,t-1}\right)\right]}_{\text{Realized Total Return}} + ...$$

and, given *observed/realized* rates of return, it is completely characterized. The realized return on assets is

$$\underbrace{\left(\sum_i r_t^i I_{a-1,t-1}^i - \sum_j r_t^j I_{a-1,t-1}^j\right)}_{\text{intercept and attached rates}} + \underbrace{\left(\sum_i r_t^i \lambda_{t-1}^i - \sum_j r_t^j \phi_{t-1}^j\right)}_{\text{marginal return on B}} B_{a-1,t-1}$$

where we note again that the parameters and intercept functions are timed in the same way as the underlying assets they describe, but the rates of return are timed one period forward.

The **marginal** rate we are looking for in the Euler equation is then

$$R_t^B = 1 + \bar{r}_t^B = 1 + \left(\sum_i r_t^i \lambda_{t-1}^i - \sum_j r_t^j \phi_{t-1}^j\right)$$

and $\bar{r}_t^B$ is not age dependent since the parameters ϕ and λ are not age dependent and we assume that rates of return r_t^i or r_t^j on any assets and liabilities are not age related. Note that interest rates on bank debt may well be age related but we rule that out.

Marginal return on Z If the auxiliary variable Z is endogenous there will be a marginal rate given by

$$\bar{r}_t^Z = \left(\sum_i r_t^i I_{z,t-1}^i - \sum_i r_t^j I_{z,t-1}^j\right)$$

This is the case for housing. Since bank debt is related to housing purchases, we select the housing stock $D_{a,t}$ (or housing value $V_{a,t}^D = P_t^D D_{a,t}$) as an auxiliary variable $Z_{a,t}$. As the portfolio is related to the housing stock, the choice of housing now influences the savings decision through its impact on portfolio composition and returns. Note that as the household changes its decision on housing D and on net financial assets B, the portfolio adjusts within the model as the data suggests it should. This adjustment is still exogenous as optimal portfolio composition is implicit in the estimated parameters of the portfolio structure.

As shown above the additional marginal rate is

$$\bar{r}_t^D = X_t \bar{p}_t^D \left(\sum_i r_t^i I_{d,t-1}^i - \sum_i r_t^j I_{d,t-1}^j\right)$$

and is generally non zero, unless the rate of return on assets and liabilities is the same. It is also not age dependent. This marginal rate helps characterize the user cost of housing in more detail as the household faces mortgage interest costs on the mortgage part, but opportunity costs on the non mortgage part. These opportunity costs now reflect also the change in portfolio weight on bank debt when the volume of housing changes.

The user cost of housing We derived the expression

$$p_{a,t}^{ucD} = \underbrace{\left[\frac{\partial f_{a,t}}{\partial D_{a,t}} + \frac{1}{1+\bar{r}_{t+1}^{B}} \frac{\partial f_{a+1,t+1}}{\partial D_{a,t}} - \frac{R_{a+1,t+1}^{D}}{1+\bar{r}_{t+1}^{B}} \right]}_{\text{User Cost of } D_{a,t} \text{ measured at time t.}}$$

and with

$$1+\bar{r}_{t+1}^{B} \equiv 1 + \left(\sum_i r_{t+1}^{i} \lambda_t^{i} - \sum_j r_{t+1}^{j} \phi_t^{j} \right)$$

where

$$R_{a+1,t+1}^{D} \equiv \bar{r}_{t+1}^{D} = \frac{B_t}{V_t^{D}} P_t^{D} \left(\sum_i r_{t+1}^{i} I_{d,t}^{i} - \sum_i r_{t+1}^{j} I_{d,t}^{j} \right)$$

and we have[26]

$$USER_{a,t} = \underbrace{\left[\frac{\partial f_{j,a,t}}{\partial D_{j,a,t}} + \frac{1}{1+\bar{r}_{t+1}^{B}} \frac{\partial f_{j,a+1,t+1}}{\partial D_{j,a,t}} - \frac{B_t}{V_t^{D}} P_t^{D} \frac{\left(\sum_i r_t^{i} I_{d,t}^{i} - \sum_i r_t^{j} I_{d,t}^{j} \right)}{1 + \left(\sum_i r_t^{i} \lambda^{i} - \sum_j r_t^{j} \phi^{j} \right)} \right]}_{\text{User Cost of } D_{a,t} \text{ measured at time t.}}$$

Shocks and data

Shocks. Each different asset or liability has its own return, and, in the absence of shocks to the model, realized and “expected” returns are identical. Since MAKRO is a perfect foresight model, when a shock occurs it changes the environment from one probability 1 scenario to a different probability 1 scenario. In the impact period of the shock (and only then), domestic stock returns (and only those) will differ from “expected” returns. Realized returns are always included in the budget constraint. Expected returns (which obey arbitrage conditions in the absence of shocks) are always included in the intertemporal first-order conditions.

Data. As households in MAKRO are divided in 100 age groups, it is a requirement of the data set used to calibrate households that it contains data distributed across those age groups.[27] The tasks that MAKRO will be used for also require that the sum of the wealth profiles over age correspond to the totals found in the national accounts. Such a data set was not available prior to the creation of the MAKRO life-cycle profiles.

The administrative data used to create the wealth profiles is drawn from the Statistics Denmark’s administrative data on wealth, with some additional data being drawn from the Lovmodel database. Aggregate data on wealth is drawn from national accounts. Returns are based on aggregate data and the portfolio composition implied by the created asset profiles.

[26] The parameters λ and ϕ are in the code $dvHh2dvHhx$ and, in the code the $I_{d,t}^{i}$ and $I_{d,t}^{j}$ are called $dvHh2dvBolig['asseti', t]$ (inside the aldersprofiler.gms file).

[27] Reference: Christian P. Hoeck (2020). “The creation of lifecycle profiles for households in MAKRO.”

The asset profiles are created using two steps: First, a correspondence between the administrative data and the asset structure in MAKRO is established. Most asset and liability types in MAKRO have clear correspondences to the administrative data. This includes bank debt, deposits, real estate, mortgages, and bonds. In MAKRO stocks are divided into foreign and domestic stocks, but Statistics Denmark's wealth data only contains information on the combined value of stocks. Data from the Lovmodel database is therefore used to divide the combined value of stocks into foreign and domestic stocks. In the second step, the asset and liability profiles are then scaled proportionately to match the aggregate values from the national accounts.

Rates of return are calculated based on aggregate values from the national accounts. Combining the rates of return with the created asset and liability age profiles results in age profiles for total return

2.13.3 Land and housing depreciation

The housing D the household owns is an aggregate object containing "bricks" and land. The entire stock of land is held by households inside their housing good. An intermediary buys "bricks" and buys land released from depreciated housing, packages these together and sells the resulting housing good to families. Here we make an important simplification to the model for practical reasons. As over time the exact composition of new housing in terms of bricks and land may change, so does the implicit composition in terms of bricks and land of the total housing holdings, and this affects households of different ages differently. We simplify the model by assuming that the composition of housing in terms of bricks and land is always identical for all households. This avoids having to trace two additional age specific stock variables (bricks and land) inside the household problem, and is similar to the assumption used in the labor market where the age distribution of workers is the same in every firm.

Now, inside the housing good "bricks" depreciate but land does not. Nevertheless, the depreciation rate of the housing object is still the depreciation rate of bricks, as the land associated with depreciated bricks is released and sold by the household. Therefore we account for the released land as "lost" in the normal law of motion

$$Z_{a,t} = D_{a,t} - \left(1 - \delta_t^{bricks}\right) D_{a-1,t-1}$$

and "recover" it as household revenues from land sales.

One final detail is that new land is released into the economy every period. The aggregate land variable grows exogenously and this land growth is helicopter dropped on households proportionally to their individual land holdings. In order to settle the accounting of land sales we must determine the individual land holdings relative to aggregate land.

Individual households own the following fraction of total land:

$$\left(\frac{D_{a-1,t-1} \cdot N_{a-1,t-1}}{\sum_a D_{a-1,t-1} N_{a-1,t-1}}\right) \frac{1}{N_{a-1,t-1}}$$

The term in brackets contains the fraction of total land held by the appropriate population group (a,t). The second term is 1 over the size of that population. The product of the two yields the fraction of individual land holdings. Eliminating terms this equals $D_{a-1,t-1}\Omega_t^{Land}$ where

$$\Omega_t^{Land} \equiv \frac{1}{\sum_a D_{a-1,t-1} N_{a-1,t-1}} = \frac{1}{D_{t-1}}$$

Now we are ready to determine revenues from land sales. The total quantity of land being sold is the land released by housing depreciation plus the helicopter land growth, $Land_t^{sales} = \delta_t^{bricks} Land_{t-1} + Land_t - Land_{t-1}$. Individual revenues from selling land are then given by

$$D_{a-1,t-1} \cdot \Omega_t^{Land} P_t^{Land} Land_t^{Sales}$$

This quantity is now adapted to the model in the main text by defining the object α_t^{Land}. This is given by

$$\alpha_t^{Land} = \frac{\Omega_t^{Land} P_t^{Land} Land_t^{Sales}}{P_{t-1}^D} = \frac{P_t^{Land} Land_t^{Sales}}{P_{t-1}^D D_{t-1}}$$

where α_t^{Land} is the same for all households.

A final remark regarding depreciation is in order. Housing depreciation is endogenous in the data. Maintenance investments prolong the life of a house. Such investments are choices and either consist of home production or of purchases from small to medium size service providers such as plumbers and carpenters. This is a simple model extension which we have developed and can later be included in MAKRO if necessary.

2.13.4 Mortgages and Housing in the budget constraint

Preliminaries. The law of motion for the housing stock is

$$D_{a,t} = \left(1 - \delta^d\right) D_{a-1,t-1} + Z_{a,t}$$

When we derive the budget constraint we consider the cases of positive versus negative net investment in housing since when $Z_{a,t} > 0$ we want to to impose a down-payment constraint but when $Z_{a,t} < 0$ we do not. In order to make the budget constraint easier to read define the composite variable

$$\Delta_{a,t} \equiv B_{a,t} - (1 + r_{a,t}) B_{a-1,t-1} - yDisp_{a,t} + rent_t H_{a,t}$$

We postulate the exogenous relationship for the mortgage debt stock $X_{a,t}^M$, such that mortgages are proportional to the value of the house

$$X_{a,t}^M = \mu_{a,t} p_t^D D_{a,t}$$

where $\mu_{a,t}$ is a variable which is exogenous to the household, and which we detail below. Endogenous mortgage ratios increase the size of the model and require modelling of credit relationships which is a non trivial expansion of MAKRO.

The budget constraint: positive investment in housing. Consider first the case of $Z_{a,t} > 0$. The term $M^{DP}_{a,t} > 0$ is the fraction or amount paid in cash when increasing the housing stock (the down-payment), and $m_{a,t}$ is an unspecified mortgage payment. In this case the size of the mortgage stock obeys the law of motion

$$X^M_{a,t} = \left(1 + r^{mort}_t\right) X^M_{a-1,t-1} + p^D_t Z_{a,t} - M^{DP}_{a,t} - m_{a,t}$$

The budget constraint of the household is

$$\Delta_{a,t} + p^C_t C_{a,t} = -M^{DP}_{a,t} - m_{a,t}$$
$$-\left(\tau^W_t + x_t\right) p^D_{t-1} D_{a-1,t-1} + p^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t$$

where τ^W is the wealth tax rate, x_t measures expenses in running the property, and the last term is the revenue from land sales. Now use $X_{a,t} = \mu_{a,t} p^D_t D_{a,t}$, and the laws of motion for D and X^M to get

$$\Delta_{a,t} + p^C_t C_{a,t} = -\left(1 + r^{mort}_t\right) \mu_{a-1,t-1} p^D_{t-1} D_{a-1,t-1} + \mu_{a,t} p^D_t D_{a,t}$$
$$-p^D_t D_{a,t} + p^D_t \left(1 - \delta_t\right) D_{a-1,t-1}$$
$$-\left(\tau^W_t + x_t\right) p^D_{t-1} D_{a-1,t-1} + p^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t$$

The budget constraint: negative investment in housing. Consider now the case of $Z_{a,t} < 0$. The budget constraint of the household does not have a down payment fraction but rather keeps the entire proceeds of the net sale

$$\Delta_{a,t} + p^C_t C_{a,t} = -p^D_t Z_{a,t} - m_{a,t}$$
$$-\left(\tau^W_t + x_t\right) p^D_{t-1} D_{a-1,t-1} + p^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t$$

Since none of the revenues are used to pay down the mortgage, the size of the mortgage stock obeys the law of motion

$$X^M_{a,t} = \left(1 + r^{mort}_t\right) \cdot X^M_{a-1,t-1} - m_{a,t}$$

When we put the two together we obtain exactly the same as above. There is no asymmetry in the problem. Of note is also the fact that the mortgage payment $m_{a,t}$ disappears entirely from the problem.

The f object. Reorganizing terms yields the f object we use in the main text.

$$f\left(D_{a,t}, D_{a-1,t-1}\right) = \left(1 - \mu_{a,t}\right) P^D_t D_{a,t}$$
$$+\left\{\left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x_t - \frac{P^D_t}{P^D_{t-1}} \left(1 - \delta^d_t\right) - \alpha^{Land}_t\right\} P^D_{t-1} D_{a-1,t-1}$$

No transaction costs. Proper aggregation of non-convexities at the micro level, such as fixed costs of trading houses, is necessary for an accurate description of aggregate behavior.[28] Given the constraints imposed by the GAMS software and by the size of the model introducing these non differentiabilities excessively increases the computational burden. In the absence of an endogenous trade-off between renting and owning we leave such adjustment costs out of the problem, and proxy for them through the reference housing value inserted into the utility function.

2.13.5 The Mortgage Ratio μ.

Mortgages are proportional to the value of the house

$$X_{a,t}^{M} = \mu_{a,t} P_t^D D_{a,t}$$

where $\mu_{a,t}$ is exogenous to the household and is given by

$$\mu_{a,t} = \tilde{\mu}_{a,t} \frac{\bar{P}_{a,t}^D}{P_t^D}$$

where $\tilde{\mu}_{a,t}$ is a calibration object exogenous to the model.

The reference price $\bar{P}_{a,t}^D$ is a function of current and past prices of the form

$$\bar{P}_{a,t}^D = \Gamma_{a,t}^p P_t^D + \left(1 - \Gamma_{a,t}^p\right) \bar{P}_{a-1,t-1}^D$$

The factor $\Gamma_{a,t}^p$ is a measure of new borrowing. A simple measure is the ratio of current investment over final stock

$$\Gamma_{a,t}^p = \frac{Z_{a,t}}{D_{a,t}} = \frac{D_{a,t} - \left(1 - \delta^d\right) D_{a-1,t-1}}{D_{a,t}}$$

In this way, for the first age of economic life when houses are bought, $\Gamma_{a,t}^p = 1$, implying all mortgages are new and subject to the current price. This number $\Gamma_{a,t}^p$ is bounded above by 1 and since D is always positive it has a finite lower bound. Younger agents are much more subject to the variation in house prices than older ones. The ability to finance through a mortgage therefore varies with house prices. Given $\Gamma_{a,t}^p$ the ratio

$$\mu_{a,t} = \tilde{\mu}_{a,t} \frac{\bar{P}_{a,t}^D}{P_t^D} = \tilde{\mu}_{a,t} \left(\Gamma_{a,t}^p + \left(1 - \Gamma_{a,t}^p\right) \frac{\bar{P}_{a-1,t-1}^D}{P_t^D} \right)$$

falls at impact with an increase in house prices. The household can mortgage more as prices increase, since $X_{a,t}^M = \mu_{a,t} P_t^D D_{a,t}$ increases with the house price keeping all else constant, but less than proportionally. Leverage ratios fall with house price increases. But since $\Gamma_{a,t}^p$ changes endogenously as housing decisions react to house prices the mortgage ratio is more reactive. Leverage ratios fall slightly more with house price increases if investment falls. The exact effect

[28] See the entire literature on firm investment with non convexities. Specific examples are Cooper and Adda (2000) on cars, Li, Liu, Yang, and Yao (2016) on housing, and Ampudia, Cooper, LeBlanc and Zhu (2019) on financial portfolio adjustment.

depends on how persistent the increase in prices is, which affects the investment decision. A temporary increase in house prices should trigger a strong fall in investment. A permanent one not necessarily so. We use a slightly more general way of writing this factor as follows

$$\Gamma^{p}_{a,t} = (1-\phi) + \phi \frac{D_{a,t} - \left(1-\delta^{d}\right) D_{a-1,t-1}}{D_{a,t}} = 1 - \phi\left(1-\delta^{d}\right)\frac{D_{a-1,t-1}}{D_{a,t}}$$

as it allows for a degree of control over the influence of the endogenous housing decision on the mortgage ratios.

2.13.6 Aggregation

Due to migration flows, population obeys

$$N_{a,t} = s_{a-1,t-1} N_{a-1,t-1} + I_{a,t} - E_{a,t}$$

While in the data immigrants and emigrants are different from the average household in most respects, the model is nevertheless bound by the necessity to fit all agents into an average that can be replicated.[29] The household model has one dimension of heterogeneity which is age. Any additional heterogeneity is eliminated.

The goal is then to generate average quantities of assets B, housing D, consumption C, and employment that encompass residents and migrants in an internally consistent way. In the labor market chapter we detail the assumptions and mechanics needed to generate average employment, and here we detail the aggregation of assets and housing. We therefore assume that migrants carry with them the necessary assets to appropriately fit the resulting average. That is not the only assumption required, so we first work through an aggregation example without housing, and then replicate it with added housing.

Aggregation without housing. We **first** detail the budget constraint of three groups of households. Surviving residents $s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}$ have the budget constraint where $x_{a,t} \equiv y_{a,t} - P^{C}_{t} C_{a,t}$ and where we have added a transfer term $T^{S}_{a,t}$ (superscript S for stayers):

$$B_{a,t} - x_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} + T^{S}_{a,t}$$

The people who leave, $E_{a,t}$ receive/pay a transfer $T^{E}_{a,t}$ before leaving. They do not earn income, consume or buy end-of-period assets in the country this period so that their constraint is simply a definition of the amount $M_{a,t}$ they carry abroad:

$$M_{a,t} \equiv (1 + r_{a,t}) B_{a-1,t-1} + T^{E}_{a,t}$$

Finally, the people who enter the country, $I_{a,t}$ receive/pay a transfer $T^{I}_{a,t}$ after arrival. This transfer and the assets A they bring into the country allow them to make identical choices to

[29] Non migrants are also heterogeneous and yet only their average by age is in the model.

those of the first group (the stayers):

$$B_{a,t} - x_{a,t} = A_{a-1,t-1} + T^I_{a,t}$$

The **next** step is to define some properties we need to impose. We choose to impose a zero net effect on the balance of payments. This implies the following relationship between households entering and exiting the country:

$$A_{a-1,t-1} = \frac{E_{a,t}}{I_{a,t}} M_{a,t} = \frac{E_{a,t}}{I_{a,t}} \left(1 + r_{a,t}\right) B_{a-1,t-1} + \frac{E_{a,t}}{I_{a,t}} T^E_{a,t}$$

We then impose that the transfer flows have zero net sum. This implies:

$$\left[s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}\right] T^S_{a,t} + E_{a,t} T^E_{a,t} + I_{a,t} T^I_{a,t} = 0$$

and finally we impose equality of stayers and new arrivals which implies

$$T^I_{a,t} = \left(1 + r_{a,t}\right) B_{a-1,t-1} + T^S_{a,t} - A_{a-1,t-1}$$

Now we can aggregate to obtain the budget constraint of the population, $N_{a,t}$, at each age. This is done in the following steps. Add surviving residents $s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}$ plus incomers

$$N_{a,t}\left[B_{a,t} - x_{a,t}\right] = \left[s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}\right]\left[\left(1 + r_{a,t}\right) B_{a-1,t-1} + T^S_{a,t}\right] + I_{a,t}\left[A_{a-1,t-1} + T^I_{a,t}\right]$$

replace the zero balance of payments relationship

$$N_{a,t}\left[B_{a,t} - x_{a,t}\right] = \left[s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}\right]\left[\left(1 + r_{a,t}\right) B_{a-1,t-1} + T^S_{a,t}\right]$$
$$+ I_{a,t}\left[\frac{E_{a,t}}{I_{a,t}}\left(1 + r_{a,t}\right) B_{a-1,t-1} + \frac{E_{a,t}}{I_{a,t}} T^E_{a,t} + T^I_{a,t}\right]$$

replace T^I with the equality of stayers and incomers and collect terms to obtain

$$B_{a,t} - x_{a,t} = \left(1 + r_{a,t}\right) B_{a-1,t-1} + T^S_{a,t}$$

Finally, determine the transfer payments T. If we put together the relationships on zero balance of payments, zero net transfers, and the equality of stayers and new arrivals, we obtain that

$$T^S_{a,t} = -\frac{\left(I_{a,t} - E_{a,t}\right)}{N_{a,t}}\left(1 + r_{a,t}\right) B_{a-1,t-1}$$
$$T^I_{a,t} = \left(I_{a,t} - E_{a,t}\right)\left[\frac{1}{I_{a,t}} - \frac{1}{N_{a,t}}\right]\left(1 + r_{a,t}\right) B_{a-1,t-1} - \frac{E_{a,t}}{I_{a,t}} T^E_{a,t}$$

and using this we can rewrite the aggregate budget constraint as

$$B_{a,t} - x_{a,t} = \left(1 + r_{a,t}\right) B_{a-1,t-1} \Gamma_{a,t}$$

where

$$\Gamma_{a,t} = \frac{s_{a-1,t-1} N_{a-1,t-1}}{N_{a,t}}$$

One last detail is that the transfer $T^E_{a,t}$ is arbitrary and can be set at zero. These transfers are artificial constructions used to ensure new arrivals and surviving residents can make the same exact decisions subject to a balance of payments constraint which in this case is zero. They are added to the problem as a lump sum which is taken as exogenous by the household and do not affect marginal decisions. Furthermore, as all agents make identical choices within age, the only budget constraint that needs to be satisfied is the aggregate one, namely

$$B_{a,t} - x_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} \Gamma_{a,t}$$

Aggregation with housing. Surviving residents $s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}$ have the budget constraint

$$B_{a,t} - x_{a,t} + (1 - \mu_{a,t}) P^D_t D_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1} + T^S_{a,t}$$

where

$$\chi_{a,t} \equiv \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x_t - \frac{P^D_t}{P^D_{t-1}} \left(1 - \delta^d_t\right) - \alpha^{Land}_t$$

The people who leave, $E_{a,t}$ do not consume or work in the country, and sell their houses so that $Z_{a,t} = -(1 - \delta_t) D_{a-1,t-1} < 0$ and take their assets and proceedings abroad. Because they are downsizing, their budget constraint does not have a down payment fraction but rather keeps the entire proceeds of the sale. The final mortgage payment m liquidates the outstanding mortgage so that we obtain

$$M_{a,t} \equiv (1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1} + T^E_{a,t}$$

and where $M_{a,t}$ is the amount these agents will take abroad.

The people who enter the country, $I_{a,t}$ have $Z_{a,t} = D_{a,t} > 0$ and they earn their income and consume here while they bring assets $A_{a-1,t-1}$ from abroad. Their housing expenditure is $(1 - \mu_a) P^D_t D_{a,t}$. Their full budget constraint is

$$B_{a,t} - x_{a,t} + (1 - \mu_{a,t}) P^D_t D_{a,t} = A_{a-1,t-1} + T^I_{a,t}$$

We now impose the three conditions from above. Once again the zero balance of payments effect is defined as

$$A_{a-1,t-1} = \frac{E_{a,t}}{I_{a,t}} M_{a,t}$$

and the property that the transfer flows have zero net sum is also as above:

$$[s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}] T^S_{a,t} + E_{a,t} T^E_{a,t} + I_{a,t} T^I_{a,t} = 0$$

and finally the equality of stayers and new arrivals now implies

$$T^I_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1} + T^S_{a,t} - A_{a-1,t-1}$$

Reproducing the same steps performed above we will find that we can set $T^E_{a,t} = 0$ and that we obtain the following objects. The aggregate budget constraint

$$B_{a,t} + (1-\mu_{a,t})\, P_t^D D_{a,t} = y_{a,t} - P_t^C C_{a,t}$$

$$+ \left[(1+r_{a,t})\, B_{a-1,t-1} - \chi_{a,t} P_{t-1}^D D_{a-1,t-1} \right] \Gamma_{a,t}$$

and the transfer

$$T_{a,t}^S = -\frac{(I_{a,t} - E_{a,t})}{N_{a,t}} \left[(1+r_{a,t})\, B_{a-1,t-1} - \chi_{a,t} P_{t-1}^D D_{a-1,t-1} \right]$$

and again the only budget constraint that needs to be satisfied is the aggregate one.

It is useful to emphasize here that cohort aggregation, which is used extensively in the code, brings in the factor $\Gamma_{a,t}$ attached to the lagged stocks $B_{a-1,t-1}$ and $D_{a-1,t-1}$. In most of the text here this does not show since we are concerned with the decisions of individual households.

2.13.7 Housing intermediary

Households buy houses from a particular intermediary agent. The purpose of having this agent is to introduce land (a quasi-fixed factor) in the model. Although studies for other countries are only indicative of possible effects in Denmark, they document land as a fundamental factor affecting house prices.[30] The housing intermediary buys "bricks" from the construction sector and land from households, packages them and sells them to households as houses. An adjustment cost function is added to the housing production function in order to help fit dynamic behavior of house prices and quantities. New houses equal gross output minus adjustment costs, $Y_t^D - AC_t$. Gross output is a CES function of land (here J_t) and the "bricks" input flow from the construction sector (here labelled I_t^D). These new houses built in a given period correspond to the aggregate net investment obtained in the household problem after summing over age

$$Y_t^D - AC_t \equiv D_t - (1-\delta_t)\, D_{t-1}$$

The intermediary agent has profits given by

$$\pi_t = P_t^D \left(Y_t^D - AC_t \right) - P_t^{Land} J_t - P_t^I I_t^D$$

Accumulating the input flow from the construction sector (I_t^D) as an investment quantity yields an accounting measure of a "stock of construction capital" embodied in owner-occupied houses:

$$K_t^D = K_{t-1}^D (1-\delta_t) + I_t^D$$

where δ is the depreciation rate of the "bricks" or buildings (as opposed to land) part of the house. To get a measure of the total stock of capital in housing consistent with the data, we

[30] Davis, M. A., and Heathcote, J. (2005). Housing and the Business Cycle. International Economic Review, Vol. 46, No. 3, pp. 751-784. Davis, M. A. and Heathcote, J. (2007). The price and quantity of residential land in the United States. Journal of Monetary Economics, vol. 54(8), pp. 2595-2620.

add an exogenous stock of rental housing. This combined stock is the main input in the synthetic housing production sector present in the input/output data, and which is described in the chapter on private production.

We specify the adjustment cost function as

$$AC_t = \frac{\gamma}{2} I_t^D \left(\frac{I_t^D}{I_{t-1}^D} - \xi_t^I \right)^2$$

while the first order conditions are

$$P_t^D \left[\frac{\partial Y_t^D}{\partial J_t} \right] = P_t^{Land}$$

$$\underbrace{P_t^D \frac{\partial Y_t^D}{\partial I_t^D}}_{\text{user cost}_t} = P_t^I + P_t^D \frac{\partial AC_t}{\partial I_t^D} + \beta_{t+1} P_{t+1}^D \left[\frac{\partial AC_{t+1}}{\partial I_t^D} \right]$$

Using the user cost expression and $\frac{\partial Y_t^D}{\partial J_t} \equiv \left(\mu_J \frac{Y_t^D}{J_t} \right)^{1/E}$ and $\frac{\partial Y_t^D}{\partial I_t^D} \equiv \left(\mu_I \frac{Y_t^D}{I_t^D} \right)^{1/E}$ we can write demand functions in CES style

$$J_t = \mu_J \cdot Y_t^D \left(\frac{P_t^{Land}}{P_t^D} \right)^{-E}$$

$$I_t^D = \mu_I \cdot Y_t^D \left(\frac{\text{user cost}_t}{P_t^D} \right)^{-E}$$

Given these demand function the optimization problem solves using both of them and the CES zero profit condition

$$P_t^D Y_t^D = P_t^{Land} J_t + \text{user cost}_t \cdot I_t^D$$

The quantity q_t^I is then used to generate the stock variable q_t^k in the standard manner

$$q_t^k = (1 - \delta_t) \, q_{t-1}^k + q_t^I$$

2.13.8 Merging profits of intermediary with the household budget constraint.

Due to the addition of adjustment costs the intermediary agents have profits. As this is an auxiliary construction to introduce land into the model which does not have a correspondence in data, these profits are plugged back into the household budget constraint as transfers that do not affect the user cost and the marginal decisions of households.[31] Profits are allocated to households by age using the weight $D_{a-1,t-1}/D_{t-1}$ as illustrated in the optimizing household budget constraint.

[31] The household model is designed to capture the majority of transactions which involve already built housing. The possibility that households themselves buy land and contract the building of the house is partially captured by the intermediary and the inclusion of its profits in the budget constraint.

$$B_{a,t} = (1 + r_{a,t})\, B_{a-1,t-1} + \tilde{y}_{a,t} - p^c_t C_{a,t} - (1 - \mu_{a,t})\, P^D_t D_{a,t}$$

$$-\left\{ \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x_t - \frac{P^D_t}{P^D_{t-1}} \left(1 - \delta^d_t\right) - \alpha^{Land}_t \right\} P^D_{t-1} D_{a-1,t-1}$$

$$+ \underbrace{\frac{D_{a-1,t-1}}{D_{t-1}}}_{weight} \left\{ \underbrace{P^D_t \left[D_t - \left(1 - \delta^d_t\right) D_{t-1}\right] - P^{Land}_t J_t - P^I_t q^I_t}_{profits} \right\}$$

At this point we work on the algebra of the budget constraint to simplify the expression and reduce the computational burden. We first decompose the last line

$$\ldots \frac{D_{a-1,t-1}}{D_{t-1}} P^D_t \left[D_t - \left(1 - \delta^d_t\right) D_{t-1}\right] - \frac{D_{a-1,t-1}}{D_{t-1}} P^{Land}_t J_t - \frac{D_{a-1,t-1}}{D_{t-1}} P^I_t q^I_t$$

and eliminate the terms in $\left(1 - \delta^d_t\right)$ to obtain

$$B_{a,t} = (1 + r_{a,t})\, B_{a-1,t-1} + \tilde{y}_{a,t} - p^c_t C_{a,t} - (1 - \mu_{a,t})\, P^D_t D_{a,t}$$

$$- \left\{ \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x_t \right\} P^D_{t-1} D_{a-1,t-1} + \alpha^{Land}_t P^D_{t-1} D_{a-1,t-1}$$

$$+ P^D_t D_{a-1,t-1} \left[\frac{D_t}{D_{t-1}}\right] - \frac{D_{a-1,t-1}}{D_{t-1}} P^{Land}_t J_t - \frac{D_{a-1,t-1}}{D_{t-1}} P^I_t q^I_t$$

now use the fact that α^{Land}_t is given by

$$\alpha^{Land}_t = \frac{P^{Land}_t J_t}{P^D_{t-1} D_{t-1}}$$

and replace so that the land terms disappear completely and obtain a budget constraint with fewer terms

$$B_{a,t} = (1 + r_{a,t})\, B_{a-1,t-1} + \tilde{y}_{a,t} - p^c_t C_{a,t} - (1 - \mu_{a,t})\, P^D_t D_{a,t}$$

$$- \left\{ \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x_t - \frac{P^D_t D_t - P^I_t q^I_t}{P^D_{t-1} D_{t-1}} \right\} P^D_{t-1} D_{a-1,t-1}$$

2.13.9 The multiplier effects of leverage.

The presence of the mortgage contract generates a leverage effect in the model. Specifically, when house prices rise, $p^D_t > p^D_{t-1}$, the existing debt obligation is valued at prices p^D_{t-1} but now the equity on the house is valued at the house price p^D_t. It may be profitable to liquidate the previous mortgage, sell the house and buy a bigger house using the fact that one only has to commit a small fraction of funds because one is allowed to borrow. This mechanism is better understood if we look explicitly at the cost of housing object f in the budget constraint.

$$f\left(D_{a,t}, D_{a-1,t-1}\right) = (1 - \mu_{a,t})\, P^D_t D_{a,t}$$

$$+ \left\{ \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x_t - \frac{P^D_t}{P^D_{t-1}} \left(1 - \delta^d_t\right) - \alpha^{Land}_t \right\} P^D_{t-1} D_{a-1,t-1}$$

We can rearrange this into

$$f = \underbrace{(1-\mu_{a,t})\,P_t^D D_{a,t}}_{\text{Out of pocket: new house equity}} - \left[\underbrace{(1-\delta_t^d)\,\frac{P_t^D}{P_{t-1}^D} - \mu_{a-1,t-1}}_{\text{exisiting house equity ratio}}\right] P_{t-1}^D D_{a-1,t-1}$$

$$+\underbrace{\left\{r_t^{mort}\mu_{a-1,t-1} + \tau_t^W + x_t - \alpha_t^{Land}\right\} P_{t-1}^D D_{a-1,t-1}}_{\text{Unavoidable net carrying costs}}$$

The key feature is that an increase in house prices has a marginal effect which is not dragged down by the previous debt $\mu_{a-1,t-1}$. We have

$$\frac{\partial f}{\partial p_t^D} = \underbrace{(1-\mu_{a,t})\,D_{a,t}}_{\text{effect on new house equity}} - \underbrace{(1-\delta_t^d)\,D_{a-1,t-1}}_{\text{effect on exisiting house equity}}$$

and since $1-\delta > 1-\mu$ the cost of housing comes down when house prices increase. Therefore it is possible to buy extra housing.

Notice that there are no transaction costs which implies taking advantage of the leverage effect is costless. This potentially makes the leverage effect very powerful.

Now, this mechanism here is static. Rational agents are forward looking so they will not rush to buy more houses if prices are likely to fall in the future, which will happen if the cause of the increase in house prices is a temporary shock. As they antecipate capital losses they will dampen their current response to the price increase.

Financial accelerator

The leverage effect is **not** the financial accelerator effect of Kyotaki and Moore (1997), or of Bernanke and Gertler (1995). In fact, the mortgage contract **worsens** with an increase in house prices, $\partial\mu/\partial p < 0$, which makes it a stabilizer rather than an accelerator. An increase in house prices, even though it raises the value of your current house does not relax the mortgage financial constraint but rather tightens it. It does, however, allow the household to exploit an available (slightly worse) contract and buy more houses simply because the household now has more money and because **there exists** an available debt contract.

Yet, MAKRO does have an accelerator in the KM and BG sense. It lies inside the utility from leaving a bequest and the utility of wealth. This object is a concave function of the sum $B + p\,(1-\mu)\,D$. This combined object has an admissible lower bound. If the household is near this lower bound, an increase in house prices allows liquid wealth B to decrease, which allows households to consume more and buy more houses. The constraint has been relaxed by the house price increase, and here, buying extra housing relaxes the constraint next period also. This dynamic effect has all the hallmarks of the classic financial accelerator mechanism.

3 Firms

The economy has eight private sectors indexed by the subscript s in this chapter. These are agriculture (including fishing), construction, energy provision, extraction, housing, manufacturing (including food processing), shipping, and services (other than shipping).

Firms maximize the present discounted value of profits. Solving this problem requires both cost minimization and optimal price setting. These two problems are separated into two sub-sectors - an intermediate sub-sector producing the goods and choosing inputs optimally, and another sub-sector where retail firms buy goods from producers, set prices and sell the same goods to the final consumers. The production and the price-setting decisions are separated both in the documentation and in the code. The production problem is given in this chapter.

The model's private sector firms use labor, capital, and intermediate goods as inputs. These inputs generate output through a production function which is a CES nest with multiple levels. Capital and intermediate goods can be bought from domestic firms or imported, and labor services are bought from supplying households. The market for intermediate inputs is a spot market and the optimal decision is static. The optimal decisions for labor and capital are dynamic, and the relevant price measures are user costs derived from intertemporal first-order conditions.

The user cost of labor is derived in the labor market chapter. The user cost of capital is derived here. Given the correct user cost measures, the problem of the firm can be solved by a sequence of static cost minimization problems at every level of the CES tree. Two bottom levels of the CES tree determine input demand for intermediate inputs and investment goods from all producing sectors and from foreign or domestic sources. These levels are separated from the problem of the firm and into the input/output system of market clearing relationships by interpreting them as intermediate transformation sectors with constant returns to scale technologies. These two levels are mainly data aggregation devices although we can also interpret them as behavioral objects. They are described in the input/output chapter.

This section delivers two demand components - intermediate inputs,$R_{s,t}$, and investments, $I_{i,s,t}$ - to the input/output chapter. The rest of this chapter is organized as follows:

- Cost minimization, subsection 3.1: The production function, the CES tree, and the general cost minimization problem.
- Dynamic optimization, subsection 3.2: The dynamic optimization problem and the computation of the user cost of capital.
- Firm investments with costly external finance, subsection 3.3: A reduced form way of modeling costly external finance through the firm's discount factor.
- Finance, subsection 3.4: A brief introduction to the valuation of the firm and how we account for firm financial assets and liabilities.
- Special sector considerations, subsection 3.5: Deviations from the baseline model for the housing and extraction sectors.

3.1 Cost minimization

3.1.1 The production function

Gross output Q is produced with capital equipment, K_{iM}, energy, E, labor, L, capital structures (buildings) K_{iB}, and intermediate inputs R.

All inputs except intermediate inputs enter production in effective units which include factor utilization and in the case of labor, net of hiring costs. Effective capital inputs are denoted $K^u_{iM,s,t-1}$ and $K^u_{iB,s,t-1}$ (superscript u for utilization). Capital stocks are subject to a one period time to build which implies they are fixed in the short run (current period) although they can be used with varying intensity.

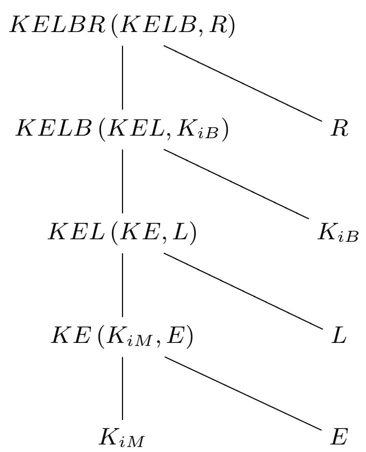
We write the general production function in sector s at time t as

$$KELBR_{s,t} = Q\left(K^u_{iM,s,t-1}, E_{sp,t}, L_{s,t}, K^u_{iB,s,t-1}, R_{sp,t}\right)$$

In what follows sector and time indices are omitted when that does not affect exposition.

3.1.2 The CES tree

Production inputs generate output through a CES nest structure where superscripts K, E, L, B, R denote equipment capital, energy, labor, structures, and intermediate inputs:



3.1.3 CES cost minimization

The optimal demand for inputs is obtained from solving a sequence of cost minimization problems at every level in the tree. As an example, the problem at the bottom branch of the upper tree is to minimize total cost $P^{KE}KE = p^K K^u + p^E E$ subject to $KE = CES\left(K^u, E\right)$. The solution to this problem is well known and yields the following objects which, with the necessary extensions and appropriate indexing, translate appropriately to all levels of the tree:[32]

[32] It is easy to show that the CES structure also captures zero elasticities of substitution. The demand functions, constraint, and CES price all go through, while we never make use of the production function, $Q^{KEL} =$

$$\text{Output} \Rightarrow KE = \left[\left(1-\omega^E\right)^{\frac{1}{\eta}} \left(K^u\right)^{\frac{\eta-1}{\eta}} + \omega^{E\frac{1}{\eta}} \left(E\right)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$\text{Derivative} \Rightarrow \frac{\partial KE}{\partial E} = \left[\omega^E \frac{KE}{E}\right]^{\frac{1}{\eta}}$$

$$\text{Demand/F.O.C.} \Rightarrow E = \omega^E KE \left(\frac{P^{KE}}{p^E}\right)^{\eta}$$

$$\text{CES Price} \Rightarrow P^{KE} = \left[\left(1-\omega^E\right)\left(p^K\right)^{1-\eta} + \omega^E \left(p^E\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

In these equations the parameters r^j are share parameters. The parameter η is the elasticity of substitution between the two inputs. The input prices p^j(not the CES prices P) are user costs except for intermediate goods where it is a CES aggregate of spot prices.

One detail to mention is that, although we show it explicitly here, the production function is never used in the numerical solution to the problem. Much like the utility function in the household problem, only its derivatives are ever needed, and they enter the problem through the demand functions shown above. The problem is solved using only the demand functions and the constraint in the form $P^{KE}KE = p^K K + p^E E$.

Solving all the problems in the tree requires knowing the correct prices of every input.

Intermediate inputs and energy in production as well as capital investments are sourced from many production sectors, both domestic and foreign, in exactly the same way as all other demand components. The prices of intermediate inputs and investments are derived in input/output chapter. Firms pay same the wage pr. unit of labor, but user costs will generally differ between sectors due to differences in labor adjustment costs etc. The user cost of labor is derived in the labor market chapter.

Finding the correct input prices of capital (and labor) requires solving a dynamic, forward-looking, optimization problem. We look at the user cost of capital in the next subsection.

3.2 Dynamic optimization

This section derives the user cost of capital. All variables and parameters in the problem generally have a sub-index (k, s, t) – that is over investment/capital type, sector and time. In this section, this index will be truncated to only t, unless otherwise explicitly stated. Given eight different private sectors and two types of capital, MAKRO has 16 appropriately indexed versions of the variables and equations described.

3.2.1 Definitions

All capital stocks have a law of motion of the following form

$min(KE, L/\mu^L)$, and of course there are no derivatives but instead optimality implies $KE = L/\mu^L = KEL$.

$$K_t = (1 - \delta_t) K_{t-1} + I_t$$

where δ_t is the depreciation rate and I_t is the investment flow.[33]

Capital stocks are subject to adjustment costs

$$AC_t = \frac{\gamma}{2} \left[\frac{K_t}{K_{t-1}} / \frac{\overline{K}_{t-1}}{\overline{K}_{t-2}} - 1 \right]^2 K_{t-1}$$

where the γ parameter controls the severity of adjustment costs and $\overline{K}_t$ is the average capital stock in the sector.[34]

Capital stocks are fixed in the short run due to one period time to build, but they can be used with varying intensity u_t. The problem of optimal capital utilization is examined below. Adjustment and utilization costs as well as vacancy costs are not explicitly measured in the data, and are modeled as unobserved lost production. Adjustment costs are subtracted from gross production, Q. Gross and net output are related by:

$$Y_t = A_{s,t} \left[KELBR_t - \sum_k AC_{k,t} \right]$$

where the index k identifies the type of capital good (equipment or structures). Note that we do not see the vacancy posting costs or labor utilization costs explicitly here, as they are modeled directly inside gross output Q and detailed in the labor market chapter. $A_{s,t}$ measures residual total factor productivity after labor-saving technological progress (embodied in the quantity of labor supplied by households), variable factor utilization, and adjustment costs in labor and capital.

3.2.2 Debt

We model the debt issued by the firm as:

$$D_t = \mu_t^D p_t^I K_t$$

where μ_t^D is a factor taken as given by the optimizing firm and which can be viewed as the result of an independent optimal corporate capital structure problem. The value of μ_t^D is then an expression of a modified Modigliani-Miller theorem. There is an implicit trade-off between bankruptcy risk and associated costs on one hand, and the gain from financing the firm at a lower rate on the other. Currently all firms in the model have the same constant debt factor $\mu_t^D = 0.4$. This way of modeling debt mirrors the way mortgage debt is modeled on our household side of the model.

---

[33] Inventory investment is assumed to be proportional to net output: $I_{invt,sp,t} = \mu_{sp,t}^{Invt} Y_{sp,t}$. It does not accumulate or contribute to production. It is just a drain on resources in order to match the model with national accounts data, where it is small fraction of total spending (less than 0.5 percent). Inventories are listed in the index i which identifies the three types of investments (inventories, equipment, structures): $i = \{iL, iM, iB\}$.

[34] We apply the same adjustment cost function to prices, wages, and capital stocks. In each case, we consider the object with two lags to be external to the optimizing agent. I.e. we do not take a derivative wrt. the effect two periods ahead. Internalizing this tertiary effect has made no practical difference in testing.

There are other reasons to link debt to the firm's capital stock. One significant component of firm debt is mortgage debt, which is closely related to (or collateralized by) the firm's capital stock. Corporate debt is also issued with various covenants (such as not allowing sales of installed capital), which serve as an indirect claim on the firm's equipment and structures.

3.2.3 The problem of the firm

For exposition purposes, we state the problem with a single capital stock rather than two, and we use only the time index on all variables. As we focus only on the optimal choice of capital, we also leave out many details of the labor input. The operating surplus π in a period t is

$$\pi_t = (1-\tau_t^c)\begin{pmatrix} P_t Y_t - P_t^R R_t - \left[1+\tau_t^L\right] w_t L_t \\ -\tau_t^K P_t^I K_{t-1} - T_t \\ -r_t^D \mu_{t-1}^D P_{t-1}^I K_{t-1} \end{pmatrix}$$

$$- P_t^I I_t + \tau_t^c \left(\delta_t^{Tax0} P_t^I I_t + \delta_t^{Tax} K_{t-1}^{Tax}\right) + \mu_t^D P_t^I K_t - \mu_{t-1}^D P_{t-1}^I K_{t-1}$$

$$+ q_t \left(\left[1-\delta_t\right] K_{t-1} + I_t - K_t\right)$$

$$+ q_t^{Tax} \left(\left[1-\delta_t^{Tax}\right] K_{t-1}^{Tax} + \left(\lambda_t^{Tax} - \delta_t^{Tax0}\right) P_t^I I_t - K_t^{Tax}\right)$$

where net output Y_t is given by

$$Y_t = A_t \left[KELBR\left(u_{iM,t} K_{iM,t-1}, E_t, L_t, u_{iB,t} K_{iB,t-1}, R_t\right) - AC_t\right]$$

To properly locate the elements in the CES tree above, the production function $KELBR$ here corresponds to the top of the tree and has the associated CES price P_t^{KELBR}.

The optimization price is

$$P_t = \frac{P_t^{KELBR}}{A_t}$$

The first block of the surplus expression in curved brackets lists elements affected by corporate taxes τ_t^c. It contains net output, minus expenses on intermediate goods and on labor costs. τ_t^K and τ_t^L are net input taxes (such as property taxes, vehicle taxes, or payroll subsidies). Finally there are lump sum taxes T_t and interest payments on debt r_t^D, all of which reduce the taxable income (corporate tax) generated by the firm.

After these we have the nominal investment cost $P_t^I I_t$, the value of the tax deduction from capital depreciation, revenues (expenses) from increases (reductions) in corporate debt, and finally, the Lagrange multiplier (Tobin's q) and its attached law of motion for capital (the standard one), and the Lagrange multiplier and its attached law of motion for tax (or book) value capital. The tax value of capital is the nominal object, K_t^{Tax}.

3.2.4 First-order conditions

The discount factor between time t and time $t+1$ is $\beta_{t+1} = \frac{1}{1+r_{t+1}}$. The required rate of return, r_t, is taken as given by the firm and may vary across sectors if investors demand different risk premia to hold equity in different sector. In subsection 3.3, we discuss the introduction of costly external finance in the firm's problem, modeled so that its effect is contained in an augmented discount factor.

The optimal choice of labor is detailed in the labor market chapter. The first-order condition for capital utilization is discussed in section (3.2.5) below. The first-order conditions for energy, E_t, and intermediate goods, R_t, are simply $P_t^{KE}\frac{\partial KE_t}{\partial E_t} = P_t^E$ and $P_t\frac{\partial KELBR_t}{\partial R_t} = P_t^R$ and, since this optimization is static, it is already obtained in the CES cost minimization problem. The first-order condition for investment, I_t, which isolates Tobin's q, is

$$q_t = P_t^I \left(1 - \tau_t^c \delta_t^{Tax0} - \left(\lambda_t^{Tax} - \delta_t^{Tax0}\right) q_t^{Tax}\right)$$

The first-order condition for the book/tax value of capital, K_t^{Tax} , is

$$q_t^{Tax} = \beta_{t+1}\tau_{t+1}^c \delta_{t+1}^{Tax} + \beta_{t+1}\left(1 - \delta_{t+1}^{Tax}\right) q_{t+1}^{Tax}$$

where δ_t^{Tax} is the tax deductible depreciation rate, δ_t^{Tax0} is the immediate tax deductible rate from investments and λ_t^{Tax} is the total share of investments that are tax deductable over time[35]. This Lagrange multiplier is given by a Bellman equation which computes the present discounted value of all future tax deductions.

The first-order condition for capital, K_t, is

$$\begin{aligned}\left(1-\tau_t^c\right) P_t \frac{\partial Y_t}{\partial K_t} + \frac{1-\tau_{t+1}^c}{1+r_{t+1}} P_{t+1} \frac{\partial Y_{t+1}}{\partial K_t} &= q_t - \frac{1-\delta_{t+1}}{1+r_{t+1}} q_{t+1} \\ &+ \frac{1-\tau_{t+1}^c}{1+r_{t+1}} \tau_{t+1}^K P_{t+1}^I \\ &- \frac{1-\tau_{t+1}^c}{1+r_{t+1}} \left[\frac{r_{t+1}}{1-\tau_{t+1}^c} - r_{t+1}^D\right] \mu_t^D P_t^I\end{aligned}$$

From the derivative of net output we isolate the user cost of capital

$$P_{t+1}\frac{\partial Y_{t+1}}{\partial K_t} \equiv \underbrace{P_{t+1}\frac{\partial Q_{t+1}}{\partial\left(u_{t+1}K_t\right)}u_{t+1}}_{P_{t+1}^K:\text{user cost of } K_t} - P_{t+1}\frac{\partial AC_{t+1}}{\partial K_t}$$

[35] This rate is lower than unity for buildings and has been greater than unity for machinery in periods.

where u_t is the utilization rate in period t ($u_t K_{t-1}$ is the effective capital input in period t). The last term measures how an increase in K_t, decided in period t, lowers adjustment costs in period $t+1$.

We insert the user cost definition in the first order condition for capital to get

$$\begin{aligned} P^K_{t+1} = & \frac{1+r_{t+1}}{1-\tau^c_{t+1}} q_t - \frac{1-\delta_{t+1}}{1-\tau^c_{t+1}} q_{t+1} \\ & + \tau^K_{t+1} P^I_{t+1} \\ & - \left[\frac{r_{t+1}}{1-\tau^c_{t+1}} - r^D_{t+1} \right] \mu^D_t P^I_t \\ & + \frac{1+r_{t+1}}{1-\tau^c_{t+1}} \left(1-\tau^c_t\right) P_t \frac{\partial AC_t}{\partial K_t} \\ & + P_{t+1} \frac{\partial AC_{t+1}}{\partial K_t} \end{aligned}$$

Some intuition is immediate. The corporate tax rate raises the user cost of capital by the factor $1/(1-\tau^c)$. Having corporate debt reduces the user cost of capital when the cost of this debt r^D_{t+1} is lower than the cost of equity funding r_{t+1}. And not surprisingly, taxes on capital τ^K_t raise the user cost.

3.2.5 Factor utilization

Intuition Factor utilization is added to the model to help generate pro-cyclical value added per worker. In order to counter diminishing returns across all factors of production due to capital rigidity, it is necessary to compensate with a mechanism that increases total factor productivity. The equations used are flexible adaptations of the following idea. Let the firm have gross output given by a function of the type $Q_t = Q\left(u_t X_t\right)$ with a generic first-order condition for optimal choice of X_t given by

$$P_t \frac{\partial Q_t}{\partial \left(u_t X_t\right)} u_t = P^X_t$$

where P^X_t is the user cost of X. Utilization is then associated with an auxiliary stock variable x which obeys the following law of motion $x_t = \left(u_t - 1\right) X_t + \lambda_t x_{t-1}$ where $\lambda_t = \bar{u}^{1/\eta}_{t-1}/\beta_t$, and where $\bar{u}_t$ is an externality term which in equilibrium equals u_t. This law of motion has a steady state solution at $x=0$ and $u=1$. We replace the choice of u_t with the choice of the stock x_t and impose the limit condition $\lim_{t\to\infty} x_t = 0$. Imposing symmetric equilibrium on the externality term, and lagging the expression one period, the resulting dynamic first-order condition is

$$u_t = \left(\frac{P^X_t/u_t}{P^X_{t+1}/u_{t+1}}\right)^{\eta^u} = \left(\frac{P^X_t/u_t}{P^X_{t+1}/u_{t+1}}\right)^{\eta^u} \left(\frac{P^X_{t-1}/u_{t-1}}{P^X_t/u_t}\right)^{-\eta^u} u_{t-1}$$

Capital We generalize the derived expression with an additional parameter, λ^u, to get the expression for utilization for each type of capital:

$$u_{k,t} = \left(\frac{P^K_{k,t}/u_{k,t}}{P^K_{k,t+1}/u_{k,t+1}}\right)^{\eta^u_k} \left(\frac{P^K_{k,t-1}/u_{k,t-1}}{P^K_{k,t}/u_{k,t}}\right)^{-\eta^u_k} (u_{k,t-1})^{\lambda^u}$$

The parameter η^u controls the degree to which capital utilization varies and is one of the key parameters that we set in order to match MAKRO's impulse responses to a number of empirical models. Setting $0 < \lambda^u < 1$ adds persistence to changes in utilization and is likewise used to match empirical IRFs.

Labor Although we do not look at the optimal choice of labor in this chapter it is convenient to discuss its utilization rate here. Labor utilization is closer to the idea of effort. The object L contains many components and we use the term P^L which is the marginal product $P_t \frac{\partial KELBR_t}{\partial L}$. In the $Q(u \cdot X)$ model above the object X equals L/u, with the resulting first-order condition

$$P^L_t \frac{\partial L_t}{\partial u_t} \frac{\partial u_t}{\partial x_t} + \beta P^L_{t+1} \frac{\partial L_{t+1}}{\partial u_{t+1}} \frac{\partial u_{t+1}}{\partial x_t} = 0$$

and using the fact that $\partial L_t / \partial u_t = L/u$ we get

$$u_t = \left(\frac{P^L_t}{P^L_{t+1}}\right)^{\eta} = \left(\frac{P^L_t}{P^L_{t+1}}\right)^{\eta} \left(\frac{P^L_t}{P^L_{t+1}}\right)^{-\eta} u_{t-1}$$

and the generalized expression used in the model is

$$u^L_t = \left(\frac{P^L_t}{P^L_{t+1}}\right)^{\eta^{uL}} \left(\frac{P^L_{t-1}}{P^L_t}\right)^{-\eta^{uL}} \left(u^L_{t-1}\right)^{\lambda^u}$$

3.3 Firm investments with costly external finance

In the following we detail a reduced form way of modeling costly external finance through the firm's discount factor. Besides financial frictions being a realistic feature of the theory of the firm, they help generate pro-cyclical investment in line with data.

3.3.1 An illustrative example

The simplest model of the firm abstracts from costly external finance. The firm has free access to funds at an exogenous rate, r_t, which is the discount rate of its cash flows, and which is the required rate of return demanded by all investors.[36] A simple illustration of this model is the one variable example where the capital stock obeys the standard law of motion $K_t = (1-\delta) K_{t-1} + I_t$, and profits are given by revenues minus investment costs,

[36] There are also no conflicts of interest between management and ownership, and no issues of corporate control.

$\pi_t = Y(K_{t-1}) - P_t^I I_t$. The unconstrained optimal choice of capital obeys

$$\frac{\partial \pi_t}{\partial K_t} + \beta_{t+1} \frac{\partial \pi_{t+1}}{\partial K_t} = 0$$

where $\beta_{t+1} = 1/(1 + r_{t+1})$ is the discount factor which is exogenous to the firm and is not affected by how much the firm chooses to invest.

However, this is an incomplete model since financing the firm's activity is a complex process. When firms finance their activity, the immediate source of funds is retained earnings or internal finance. This is often not enough and firms also interact with banks, issue corporate debt, and raise or buy back equity. One common characteristic of all sources of outside finance is that it is costly and therefore, we want to extend the model of the firm by defining this cost. Before that, in order for costly external finance to matter the firm must be in need of funds: it must be constrained.

The simplest model of costly outside finance adds a constraint $\pi_t \geq \bar{\pi}$. Outside finance beyond this threshold is infinitely costly. With a Lagrange multiplier ξ the optimal decision obeys

$$\frac{\partial \pi_t}{\partial K_t} + \left[\beta_{t+1} \frac{1 + \xi_{t+1}}{1 + \xi_t}\right] \frac{\partial \pi_{t+1}}{\partial K_t} = 0$$

where if $\pi_t < \bar{\pi}$ then $\xi_t > 0$ (and large enough). The firm can raise cash freely up until the threshold and after that the cost of outside finance is prohibitive. The optimality condition shows that this financial friction is fully captured in the generalized discount factor. While this is not a general property of models of firm finance, we choose our model so that we have it. When the constraint binds, this generalized discount factor is lower (the future becomes less relevant). This is intuitive as the effect of costly external finance is to reduce investment. However, that is not the only effect of costly external finance on the firm. In fact, how this constraint moves is more interesting over the business cycle. Although the existence of costly external finance acts to *reduce* investment, as long as the constraint is normally binding what matters is how the constraint tightens and relaxes. These effects through the constraint help propagate the business cycle.

In MAKRO we model the costs of external finance assuming that they generally bind. We do this by using a differentiable cost function as in Gomes (2001) as a flexible way of modeling the above Lagrange multiplier. His model is as follows. Dividends are

$$d_t = \pi_t - \xi(m_t)$$

with the function ξ having value zero for $m \leq 0$ and a positive increasing function for $m > 0$ where $m_t = -\pi_t = P_t^I I_t - Y_t$, and it is easy to verify that it yields the same first-order condition

$$\frac{\partial \pi_t}{\partial K_t} + \left[\beta_{t+1} \frac{1 + \frac{\partial \xi_{t+1}}{\partial m_{t+1}}}{1 + \frac{\partial \xi_t}{\partial m_t}}\right] \frac{\partial \pi_{t+1}}{\partial K_t} = 0$$

where the Lagrange multipliers are replaced with the derivatives of the cost function.

3.3.2 The financial constraint in MAKRO

The firm maximizes

$$V_t = m_t - \xi_t + \beta_{t+1} V_{t+1}$$

where m_t is the free cash flow of the firm (excluding financial assets). The positive valued function $\xi_t \equiv \xi(m_t - \overline{m}_t) \geq 0$ models the cost of financial frictions. The first-order condition with respect to a given input X_t is

$$\frac{\partial m_t}{\partial X_t} + \hat{\beta}_{t+1} \frac{\partial m_{t+1}}{\partial X_t} = 0$$

where the augmented discount factor $\hat{\beta}_{t+1}$ is given by

$$\hat{\beta}_{t+1} = \beta_{t+1} \frac{1 - \partial \xi_{t+1} / \partial m_{t+1}}{1 - \partial \xi_t / \partial m_t}$$

We can rewrite the augmented discount factor as $\hat{\beta}_{t+1} = 1/(1 + r_{t+1} + r^p_{t+1} + r^{\xi}_{t+1})$ by defining a premium r^{ξ}_{t+1} associated with the financial constraint.

We specify the financial friction function ξ as a differentiable symmetric function with derivative given by the hyperbolic tangent function[37]

$$\frac{\partial \xi_t}{\partial m_t} \equiv \mu \tanh\left(\kappa \left[m_t - \overline{m}_t\right]\right)$$

where $\lim_{m_t \to \infty} \partial \xi_t / \partial m_t = \mu$ and $\lim_{m_t \to -\infty} \partial \xi_t / \partial m_t = -\mu$. Numerically, with $\mu = 2\%$ the firm pays a 2 pct. fee for outside finance ($m_t < \overline{m}_t$). The parameter $\kappa > 0$ controls the slope of the derivative around $m_t - \overline{m}_t = 0$.

3.3.3 Homogeneity

The function $\xi_{s,t}$ is sector specific with sector subscript s, and is approximately homogeneous if $\overline{m}_{s,t} = 0$ and with high enough κ. To preserve long run homogeneity with a free cash flow target different from zero we let $\overline{m}_{s,t}$ endogenously adjust as

$$\overline{m}_{s,t} = \gamma \overline{m}_{s,t-1} + (1 - \gamma) m_{s,t} - \epsilon_{s,t}$$

where γ controls the speed of adjustment and $\epsilon_{s,t}$ is calibrated to set $\overline{m}_{s,t} = m_{s,t}$ in the baseline forecast. We calibrate $\kappa_{s,t} = \frac{\hat{\kappa}}{K_{s,t}}$ (other measures of sector size can be used) to ensure similar behavior across sectors. For strict homogeneity $\kappa_{s,t}$ would need to endogenously adapt to changes in $K_{s,t}$, but this is unlikely to matter in practice as $m_{s,t} - \overline{m}_{s,t}$ converges to zero.

[37] The function $tanh(x) = \left(e^x - e^{-x}\right) / \left(e^x + e^{-x}\right)$ is well defined for the entire support of x. This makes the primitive function ξ a two-sided penalty function for missing profit targets and can be viewed as the cost of issuing $(m_t < \overline{m}_t)$ or re-purchasing $(m_t > \overline{m}_t)$ equity.

It is useful to define the 'habit adjusted' free cash flow $\hat{m}_{s,t}$:

$$\hat{m}_{s,t} \equiv m_{s,t} - \overline{m}_{s,t} = \gamma \left[m_{s,t} - m_{s,t-1} + \hat{m}_{s,t-1} \right] + \epsilon_{s,t}$$

Rearranging the augmented discount factor we get an expression for the discount rate premium associated with the financial constraint

$$r^{\xi}_{s,t+1} = \left[1 + r_{t+1} + r^{p}_{t+1} \right] \left[\frac{1 - \mu_s \tanh\left(\kappa_s \hat{m}_{s,t}\right)}{1 - \mu_s \tanh\left(\kappa_s \hat{m}_{s,t+1}\right)} - 1 \right]$$

3.3.4 Acceleration

The financial constraint model we use is closer to the Gomes (2001) model than to collateral acceleration models such as Bernanke and Gertler (1989). The factors relaxing the constraint in good times come from the expansion of revenues as prices increase and from the increased production ability as capital accumulates. There is no additional contribution from an improvement in collateral values or from any additional relaxation of the constraint from a goodwill effect arising from extra revenues.[38]

3.4 Finance

Production firms hold and manage liquid assets (cash, credit lines, liquid assets) since, due to financial frictions, cash inside the firm is not identical to cash outside it. These frictions are currently modeled parsimoniously in the production model and so the presence of financial assets in the firm's balance sheet is dealt with in reduced form and separated from the firm's optimization problem.

Financial services also account for a significant share of total services. We do not have a model of the financial firm, and therefore there is no theory of endogenous optimal decision making for these agents, but we do have basic principles of financial arbitrage which we apply to include financial stocks and flows in the MAKRO model. These assets must be adequately accounted for in order to fit the national accounts data (Nationalregnskabet) and make up a separate entity from production firms.

The data contains both financial and production components of firm valuation. Financial assets are valued at current market price[39]. We now provide a brief introduction to financial valuation and accounting. More detailed descriptions are contained in the appendix.

[38] These mechanisms are discussed in recent work. See Lian and Ma (2020) and Drechsel (2021) and references therein.

[39] The value of cash and other low-return instruments outside the firm would be higher than the discounted nominal value of their returns inside the firm. This is corrected by their convenience yield: this is the exact difference between the nominal yield the asset generates and the required rate on equity. Keynes used the idea of convenience yield in his money demand function. See also Del Negro et al. (2017), Safety, Liquidity, and the Natural rate of Interest, Brookings Papers, Spring 2017.

3.4.1 Valuation algebra

We divide the value of the firm, V_t, into two separate objects: 1) The value of the main operations of the firm V^x, and 2) the value of net financial assets not related to the main operations, B^x_t.

$$V_t = V^x_t + B^x_t$$

The value of operations is the discounted value of the cash flows that result from the endogenous optimal decisions of the firm:

$$V^x_t = \frac{\pi_{t+1} + V^x_{t+1}}{1 + r^E_{t+1}}$$

The required return on equity is the sum of the bond rate and an equity risk premium and differs from the discount rate used by the firm in its optimal production decision.

Net financial assets consist of a several asset classes (index A) and two types of debt (index L for liabilities). All the portfolio components are valued at their market price. For example, financial firms hold a large number of mortgage bonds that we value at their market price rather than discounting their associated interest rate payments with the required return on equity. The debt associated with capital, $\mu^D_t P^I_t K_t$, is included in the value of main operations and is therefore not included in net financial assets.

$$B^x_t = \underbrace{\sum_A B_{A,t}}_{\text{financial assets}} - \left[\underbrace{\sum_L B_{L,t}}_{\text{total debt}} - \underbrace{\mu^D_t P^I_t K_t}_{\text{debt from operations}}\right]$$

We assume that the firm holds financial assets proportional to the value of their main operations:

$$B^x_t = \left(\sum_A \mu^B_{A,t} - \sum_L \mu^B_{L,t}\right) V^x_t$$

We can now write a budget constraint of the firm which ties together net financial assets with the cash flow from main operations, dividends paid and finally revenue from new equity issues (buybacks if negative):

$$B^x_t = \left(1 + [1 - \tau^c_t]\, r^{Bx}_t\right) B^x_{t-1} + \pi_t - div_t + issues_t + \underbrace{\epsilon_t}_{\text{other transfers}}$$

where r^{Bx}_t is the rate return on net financial assets before taxes.

We assume that firms follow a constant dividend rate policy such that dividends are proportional to the total value of shares,

$$div_t = r_t^{div} V_{t-1}$$

implying equity issues are endogenous and determined residually by the budget constraint.[40] The change in share price is then given by

$$r_t^{omv} = \frac{V_t - issues_t}{V_{t-1}}$$

Note that the rate of return on equity $\left(r^{div} + r^{omv}\right)$ is unaffected by the dividend policy and equity issuance. I.e., if the firm issues equities and uses the money to buy financial assets, the value of shares including financial assets, V_t, increases by exactly the same amount and share prices do not change. Similarly, when the firm pays dividends, their value decreases by the same amount.

3.4.2 Financial accounting terms and corporate taxes

In order to define the scope of corporate taxation τ_t^c we make a detour to discuss how EBITDA (Earnings Before Interests, Taxes, Depreciation and Amortization) , and how EBT (Earnings Before Tax), translate into the income the cash flow from main operations, π, (which we used in the dynamic optimization problem of the firm):

$$\pi_t = \underbrace{P_t Y_t - P_t^R R_t - \left[1 + \tau_t^L\right] w_t L_t - \tau_t^K P_t^I K_{t-1} - T_t}_{\text{EBITDA}}$$

$$- P_t^I I_t + \underbrace{\mu_t^D P_t^I K_t - \left(1 + r_t^D\right) \mu_{t-1}^D P_{t-1}^I K_{t-1}}_{\text{new debt without interest tax deduction}}$$

$$- \tau_t^c \left[\underbrace{\text{EBITDA}_t^x - \delta_t^{Tax} K_{t-1}^{Tax} - r_t^D \mu_{t-1}^D P_{t-1}^I K_{t-1}}_{\text{EBT}^x}\right]$$

EBT^x consists of earnings before taxes *and* before income from net financial assets. To get the total tax base for corporate taxes we add the income from the net financial assets part of the firm:

$$\text{EBT}_t = \text{EBT}^x + r_t^{Bx} B_{t-1}^x$$

Note that the "before taxes" in EBITDA and EBT refers to corporate taxes only and not production taxes. Net production taxes consist of τ_t^L (payroll taxes and subsidies), τ_t^K (land and vehicle taxes) and T_t (other production taxes, which are treated as exogenous to the firm),

[40] The separation between dividends and equity issuance has minor tax implications, but is primarily necessary in order to match national accounting data.

which can all be considered part of the net prices of the firm's production inputs.

3.5 Special sector considerations

3.5.1 The extraction sector

The extractions sector consists of oil and gas and a small amount of gravel extraction. Output and prices from this sector are exogenous and based on forecasts from the Danish Energy Agency (Energistyrelsen). Given the exogenous output, production factors are endogenously chosen to minimize costs and modeled the same way as other sectors apart from the existence of an additional tax:

$$\tau^{c}_{\text{extraction},t} = \tau^{c}_{t} + \frac{y_{\text{extraction},t} - y^{\text{gravel}}_{t}}{y_{\text{extraction},t}} \tau^{oil}_{t}$$

As both the price and domestic output quantity is exogenous, we scale the demand for domestic extraction output proportionally to match the supply. Demand for extraction imports is scaled inversely such that the total demand for extraction does not depend on the exogenized sector output. I.e., import and domestic production of extraction are perfect substitutes, and we allocate a share of domestic output to different types of demand and use imports to satisfy residual demand.

3.5.2 The housing sector

In the national accounts, the housing sector produces housing services where the key input is a measure of the stock of houses. The sector includes both rental and privately owned housing and produces a homogeneous good priced at the rental value of housing. The rent value of owner-occupied housing is imputed based on rents of comparable rental housing.

In the model, we have an endogenous decision by households on owner-occupied housing while rental housing enters the household problem exogenously. As we do not (yet) model the rental market, both consumption and production of rental housing are exogenous. The housing stock available for rent is exogenous and assumed to depend on government-supported rental building projects.

Given that the rental part of housing is exogenous, we need to decouple it from owner-occupied housing. In order to achieve this separation, we model the production of housing services using a Leontief production function. The main input is the existing stock of housing (structures) and there is no input of capital equipment. Note that this measure of housing stock excludes land, and therefore it is not the housing measure, D, described in the household chapter. In the data, this sector accounts for services such as housing maintenance $(\approx 15\%)$, which involves labor and intermediate inputs, and other (mainly financial) services $(\approx 10\%)$, and for that reason it is organized as a production sector within the input/output structure of the economy.

These three inputs, structures, labor, and intermediate goods, generate output through a Leontief production function. The consequence is that output, employment, and intermediate goods are proportional to the capital stock measure. Unlike production in the other private sectors, there are no adjustment costs to capital, and therefore, net and gross production are the same.

Since everything is proportional to capital, we can separate all inputs and output in proportion to the fraction of structures that are owner-occupied housing (described as "bricks" in the household chapter) and the fraction which is rental housing. This is equivalent to assuming separate but identical Leontief production technologies for owner-occupied and rental housing, so they have equal amounts of intermediate goods and labor in proportion to their respective building capital stock.

Once the rental part of the data is separated, we calculate how it behaves in existing data and forecast its use of resources in the future.

Formalizing the problem Net output Y_t uses energy (E), labor (L), structures (K_{iB}), and intermediate goods (R):

$$Y_t = A_t{\cdot}KELBR_t\left(E_t, L_t, K_{iB,t-1}, R_t\right) = A_t{\cdot}\min\left(\frac{\phi_{iB}}{\phi_R}R_t, \min\left(\phi_{iB}K_{iB,t-1}, \min\left(\frac{\phi_{iB}}{\phi_L}L_t, \frac{\phi_{iB}}{\phi_L}E_t\right)\right)\right)$$

and the Leontief function implies the optimality conditions $KELBR_t = \phi_{iB}K_{iB,t-1}$, $E_t = \phi_E K_{iB,t-1}$, $L_t = \phi_L K_{iB,t-1}$, and $R_t = \phi_R K_{iB,t-1}$.

The capital stock $K_{iB,t}$ (here, the total stock of structures or "bricks"), and the associated investments, are the sum of owner-occupied and rental housing. The owner-occupied housing part is endogenous, while the rental part is an exogenous projection. The owned housing part of this stock is described in the household chapter in the section covering the housing intermediary. Output, employment, and the use of intermediate inputs are endogenous, but all of them are proportional to the stock due to the Leontief production function.

Link to the household problem As we decouple rental housing from owned housing, we introduce them differently in the household problem. The rental housing coming out of this sector is added exogenously to the household's budget constraint, while owner-occupied housing is an optimal decision. Owner-occupied housing is bought from an intermediary that takes a flow product from the construction sector, puts it on a plot of land, and sells the final combined good (bricks and land) to the household. Rental housing does not explicitly contain land.[41] If we remove the value added of land from the house bought by households, we have a stock equivalent to the rental housing capital, and the sum of the two stocks match the data from the housing services sector described here.

In the problem of the household we can find in the budget constraint a term which accounts for expenses with housing maintenance, $x_t P^D_{t-1} D_{a-1,t-1}$. The factor x_t in this term is taken as given by the household and is the fraction of the energy, labor and intermediate goods costs $\left[P^E_t\phi_E + P^L_t\phi_L + P^R_t\phi_R\right]$ we have in the problem above which is allocated to owner-occupied housing.

[41] Land is introduced in MAKRO in this specific way to improve house prices' modeling. The user cost of owner-occupied housing includes the effect of the price of land. Currently, land is not included in any other product or sector.

Finally, the household owns the stock of owner-occupied housing and pays directly for its maintenance costs. Therefore only the fraction of sector profits corresponding to rental housing can be allocated through the ownership of the firm in the household portfolio, controlling for the fact that households consume an exogenous amount of rental housing and pay rent on it.

4 Price setting

The main contribution of this chapter is explaining how output prices are determined in MAKRO. It also explains how import prices and export competing prices are determined.

The problem of the firms can be divided in two: Cost optimization and setting the optimal output price. The chapter of the firm deals with cost optimization and this chapter deals with setting the optimal output price. Cost optimization gives an optimal marginal cost per produced output unit including factor-specific production taxes, $P^{KLBR}_{s,t}$. Including non-factor-specific production taxes this gives us the (marginal) unit costs, $P^0_{s,t}$. In a perfect competition setting the optimal output price, $P_{s,t}$, would equal the unit costs, $P^0_{s,t}$.

In data observed prices are more sluggish than the ones generated by the perfect competition solution. The standard way to solve this problem is to add an intermediate layer of price setting behavior between the producing firm and the agents demanding the output so that the output prices are not identical to the unit costs.

$$\underbrace{(P_{s,t}|_{\text{all s}}, w_t, r_t)}_{\text{Input Prices}} \underbrace{\Longrightarrow}_{\text{Production}} \underbrace{P^0_{s,t}|_{i \in s}}_{\text{Unit costs}} \underbrace{\Longrightarrow}_{\text{Price Setting}} \underbrace{P_{s,t}|_{i \in s}}_{\text{Final Price}}$$

This price setting intermediate model is often modeled as a monopolistic competition problem. We also adopt that model and apply it to all private production sectors except housing.

In addition to monopolistic competition, we have price rigidity coming from adjustment costs of changing prices. Monopolistic competition alone does not generate price rigidity. It merely provides a theoretical foundation for price setting behavior, which in turn generates price rigidity.

4.1 Monopolistic competition and price rigidity

Monopolistic competition is a superstructure added to the problem of the firm, where every sector is thought of as having a continuum of firms with unit mass, each producing an individual "variety". Demand for all varieties is a standard CES aggregator with a demand elasticity, and in equilibrium the price paid by the consumer, $P_{s,t}$, is a markup over the marginal cost of production which reflects this elasticity. The equilibrium is symmetric so that in the end the unit mass of firms within a sector looks like a single representative firm.

While the unit costs $P^0_{s,t}$ is flexible, the consumer price is not, and therefore on top of this structure we add price rigidity through a cost of price adjustment similar to Kravik, Motzfeldt and Mimir (2019).[42]

[42] Kravik, Erling Motzfeldt og Yasin Mimir (2019). "Navigating with NEMO". I: p. 177.

4.2 Monopolistic Competition Model

In what follows we disregard the sector index s. Within each sector, firms are subject to monopolistic competition. In the monopolistic competition set-up all firms within each sector face the same demand elasticity, η_t, and the aggregate price over all firms in a given sector, P_t, is a CES price index.

Without price-adjustment costs P_t would be a constant markup over the marginal cost of production, P_t^0. However, prices are sticky as we assume these firms pay a quadratic adjustment cost to change them. The adjustment cost function is similar to Rotemberg (1982), but instead of the cost being applied to changes in the price level, p_t/p_{t-1}, it is applied to changes in inflation (as in Kravik, Motzfeldt and Mimir, 2019) which allows for richer dynamics.[43]

The monopolistic competition model generates the following demand aimed at the individual firm:

$$y_t^j = \left(\frac{p_t^j}{P_t}\right)^{-\eta_t} Y_t$$

where p_t^j is firm $j's$ price and Y_t is the sector's total production (net of capital adjustment costs).

In the absence of price adjustment costs firms would set their price as the following markup over marginal costs:

$$P_t^* = \left(1 + \frac{1}{\eta_t - 1}\right) P_t^0 = (1 + \theta_t^*)\, P_t^0$$

In the presence of price adjustment costs the markup relationship is more general.

4.2.1 Optimization Problem

Each firm j in this sector faces adjustment costs of changing prices given by:

$$g_t^j = \frac{\gamma_t}{2}\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}} - 1\right]^2 P_t^0 Y_t$$

Firm j in this sector solves the dynamic problem

$$V_t^j = max_{p_t^j}\left\{\left(p_t^j - P_t^0\right) y_t^j - g_t^j + \beta_{t+1} V_{t+1}^j\right\}$$

[43] [Rotemberg, Julio (1982). "Monopolistic Price Adjustment and Aggregate Output". I: Review of Economic Studies 49.4, s. 517–531.]

subject to

$$y_t^j = \left(\frac{p_t^j}{P_t}\right)^{-\eta_t} Y_t$$

and to the adjustment cost function above.

The first order condition is

$$p_t^j = \left(1 + \frac{1}{\eta_t - 1}\right) P_t^0 - \frac{1}{\eta_t - 1}\frac{p_t^j}{y_t^j}\left[\frac{\partial g_t^j}{\partial p_t^j} + \beta_{t+1}\frac{\partial g_{t+1}^j}{\partial p_t^j}\right] \tag{4.1}$$

The derivatives of the adjustment cost function are

$$\frac{\partial g_t^j}{\partial p_t^j} = \gamma_t \left(\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}}\right] - 1\right)\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}}\right]\frac{1}{p_t^j}P_t^0 Y_t$$

$$\begin{aligned}\frac{\partial g_{t+1}^j}{\partial p_t^j} &= -2\gamma_{t+1}\left(\left[\frac{p_{t+1}^j/p_t^j}{p_t^j/P_{t-1}}\right] - 1\right)\left[\frac{p_{t+1}^j/p_t^j}{p_t^j/P_{t-1}}\right]\frac{1}{p_t^j}P_{t+1}^0 Y_{t+1} \\ &= -2\frac{p_{t+1}^j}{p_t^j}\frac{\partial g_{t+1}^j}{\partial p_{t+1}^j}\end{aligned}$$

It is helpful to define the auxiliary variable

$$\begin{aligned}\Psi_t &\equiv \frac{1}{\eta_t - 1}\frac{p_t^j}{y_t^j}\frac{1}{P_t^0}\frac{\partial g_t^j}{\partial p_t^j} \\ &= \psi\left(\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}}\right] - 1\right)\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}}\right]\frac{Y_t}{y_t^j}\end{aligned}$$

where $\theta_t^* = \frac{1}{\eta_t - 1}$ is the structural markup, i.e. the markup in the absence of price rigidity, and

$$\psi \equiv \gamma_t \theta_t^*$$

Inserting the auxiliary variable in the first order condition (4.1) we get

$$p_t^j = (1 + \theta_t^*)\, P_t^{j,0} - \left[\Psi_t - \beta_{t+1} 2\Psi_{t+1}\frac{P_{t+1}^0}{P_t^0}\frac{y_{t+1}^j}{y_t^j}\right] P_t^0$$

Finally, using symmetry and the unit mass assumption we obtain the expression

$$P_t = \left[1 + \theta_t^* - \Psi_t + \beta_{t+1} 2\Psi_{t+1}\frac{P_{t+1}^0}{P_t^0}\frac{y_{t+1}^j}{y_t^j}\right] P_t^0$$

This can also be written as

$$P_t = (1 + \theta_t)\, P_t^0$$

where the endogenous markup θ_t is a function of a structural markup θ_t^* and price rigidity

$$\theta_t = \theta_t^* - \Psi_t + \beta_{t+1} 2\Psi_{t+1} \frac{P_{t+1}^0}{P_t^0} \frac{y_{t+1}^j}{y_t^j}$$

4.3 Sector considerations

The price-setting model is a filter which takes as an input unit costs which are highly volatile, and adds structure to it, generating output prices, which behaves in a more sluggish way.

When we take the price setting model to the data we assume independence of θ^* and ψ. We obtain a positive value of θ^* in the largest private sectors. But markups in the agriculture, energy, and shipping sectors are volatile and often negative during the data years - for these sectors we forecast θ^* as zero.

The extraction sector has a large positive markup, but it is hard to think of its price as being determined anywhere else than in the world market. Therefore, despite the large positive markups this is not a sector where the price setting structure is likely to apply, and we exogenize its price instead.

In the shipping sector, we also model a world price, by keeping the price of shipping exports constant relative to competing foreign prices, letting the markup adjust endogenously.

Finally, the housing sector is primarily a synthetic national accounting sector, which transforms the stock of owner-occupied houses into a consumption flow that is included in GDP etc. The price attached to the housing consumption flow is the rental housing price and not the user cost of housing. As the Danish rental market is highly regulated, we do not model the rental price as a market-clearing price. Instead, we model the rental price (which is also the accounting price for owner-occupied housing services) as a moving average of other private consumption prices.

4.4 Foreign prices

In MAKRO there are two exogenous foreign prices: The oil price, P_t^{oil}, and the foreign price excluding effects from the oil price and from rigidities, P_t^F.[44] All import prices, $P_{s,t}^M$, and export competing prices, $P_{x,t}^{XF}$, are determined on the basis of these two exogenous prices. For given P_t^{oil} and P_t^F all other foreign prices are constant. It should be noted that all foreign prices are determined for given interest rates and for given foreign imports. Correlations in oil prices, other foreign prices, the interest rate and foreign imports must be handled outside the model[45].

The import price for each sector is specified in the following way

[44] The oil price is directly observable and the foreign price excluding oil and rigidities is calibrated historically using aggregate import prices. The oil price in DKK, P_t^{oil}, is a price index that is determined by the Brent price of oil in dollars times the exchange rate between dollars and DKK. In a strictly technical sense it is not endogenous, but as the Brent price of oil in dollars and the exchange rate are exogenous variables - P_t^{oil} is unaffected by all endogenous responses in MAKRO.
[45] An explicit satellite model connecting these variables will likely be added in a future version of MAKRO.

$$P_{s,t}^{M} = \left[\mu_{s,t}^{pM}\left(P_t^{oil}\right)^{\sigma_s^{pM}}\left(P_t^{F}\right)^{1-\sigma_s^{pM}}\right]^{1-\varphi_s^{pM}}\left[P_{s,t-1}^{M}\right]^{\varphi_s^{pM}}$$

where $P_{s,t}^{M}$ is the current import price for sector s, $\mu_{s,t}^{pM}$ is an unobservable scale parameter, and P_t^{oil} is the current oil price. σ_s^{pM} is the price elasticity of oil while φ_s^{pM} measures the price stickiness of the import price in sector s. [46]

The export competing prices are determined in a similar way by

$$P_{x,t}^{XF} = \left[\mu_{x,t}^{pXF}\left(P_t^{oil}\right)^{\sigma_x^{pXF}}\left(P_t^{F}\right)^{1-\sigma_x^{pXF}}\right]^{1-\varphi_x^{pXF}}\left[P_{x,t-1}^{XF}\right]^{\varphi_x^{pXF}}$$

where $\mu_{x,t}^{pXF}$, σ_x^{pXF} and φ_x^{pXF} are scale parameters, price elasticity of oil and price stickiness for the export competing prices for exports groups x.

[46] This equation enables us to analyze three types of shocks: Oil price shocks, general foreign price shocks and shifts in $\mu_{s,t}^{pM}$. The expression consists of three unknowns: $\mu_{s,t}^{pM}, \sigma_s^{pM}, \varphi_s^{pM}$. The parameters σ_s^{pM} and φ_s^{pM} are estimated empirically in Kastrup et al. (2021) for each sector. Applying the estimates of σ_s^{pM} and φ_s^{pM}, $\mu_{s,t}^{pM}$ can be calibrated residually and forecast based on the auto-arima forecast method applied in MAKRO to calibrate scale parameters. This unobserved variable reflects all factors net of oil prices that affect the import price. Such factors could be foreign producer price, exchange rate fluctuations, changes in the degree of pricing-to-market, and markup variations of foreign firms.

5 The input/output system

The input/output (IO) matrix organizes market clearing conditions, equating demand and supply of goods and services. We follow the National Accounting classification where aggregate demand consists of intermediate inputs in production, R, private consumption, C, government consumption, G, investment, I, and exports, X. Demand is met by domestic production, Y, and imports, M. In current prices[47]:

$$Y + M = R + C + G + I + X \tag{5.1}$$

The right hand side objects in equation (5.1) are aggregated from various demand side compositions, investment differs according to the type of capital good for example. Each demand object (R,G,etc) combines inputs from potentially all sectors, and from both domestic production and imports.

The IO system can be viewed as a retailer buying domestic production and imports and selling intermediate product bundles to the demand side under perfect competition. The final demand products are produced with a nested production function. There are two nests. The lower nest is between imports, $M_{d.s,t}$, and domestic input, $Y_{d.s,t}$, from a specific sector. The upper nest is between aggregates of foreign and domestic inputs, $YM_{d.s,t}$, from the different sectors. The elasticity of substitution in all the upper nests is 0, but in the lower nest between imports and domestic production most elasticities are non-zero.

Tables 5.1 and 5.2 show the input/output structure of MAKRO in a single year. The first 9 rows represent domestic production sectors, while the next 4 rows are import sectors[48]. Each column represents a demand side: intermediate inputs and energy used in each of the 9 production sectors, 5 private consumption groups, a single type of government consumption, 3 investment types (including inventory investments), and finally 4 export groups (excluding tourism exports which are included in the private consumption groups). Energy and intermediate inputs are split by assuming that all output from extraction("udv" in the table) or energy("ene" in the table) are energy and the remaining sectors produce intermediate inputs. The numbers shown are values of the input/output combinations in current prices before taxes in 2022 (bn. DKK).

5.1 Market clearing prices

Markets really exist only at the bottom of the CES trees, and thus this is where equilibrium prices are determined. All other prices are constructed from these equilibirum prices. In MAKRO the most disaggregate production level is the sectoral level, indexed s. All output

[47] For ease of notation we omit deflators for this equation i.e. we write Y instead of $P^Y_t Y_t$ etc.
[48] The imports in MAKRO is in the current model version tied to import groups from ADAM and the mapping leaves agriculture and shipping empty (imports from housing, public production and construction are also 0 in the national account IO-tables or very close to 0). For more detail refer to the appendix on "Data and calibration".

Table 5.1

Input/output matrix (part 1/2)

		R (intermediate inputs)							E (Energy)		I (investments)		
		tje	fre	byg	lan	soe	bol	off	ene	udv	IB	IL	IM
Domestic production	Agriculture (lan)	1,1	48,9	0,1	11,1	0,0	0,0	0,4	1,4	0,0		1,1	0,2
	Construction (byg)	31,9	3,8	2,1	1,0	0,4	24,3	8,6	2,6	0,7	227,5		0,7
	Energy provision (ene)	14,0	6,3	2,4	2,2	0,0	0,5	6,5	4,3	0,1		0,4	1,0
	Extraction (udv)	0,4	0,8	1,2	0,2	0,0	0,0	0,1	6,7	2,4		0,2	0,1
	Housing (bol)	0,0											0,1
	Manufacturing (fre)	36,7	110,9	53,0	15,5	2,7	0,0	9,5	1,7	0,7		3,2	54,7
	Public (off)												22,1
	Services, other (tje)	502,3	106,5	80,2	15,5	8,5	31,9	135,1	7,8	1,8	19,1	1,4	96,9
	Shipping (soe)	13,3	2,0	0,0	0,0	3,2	0,0	1,0	0,1	0,3			0,2
Imports	Energy provision (ene)	7,7	1,3	0,9	1,7	28,3	0,0	0,6	1,5	0,1		0,2	
	Extraction (udv)	0,0	0,1		0,0		0,0	0,0	10,4	0,0		0,1	
	Manufacturing (fre)	38,3	155,0	32,6	9,6	0,3	0,0	27,4	3,6	0,7		4,4	68,6
	Services, other (tje)	155,7	33,9	26,9	0,5	150,9	0,2	18,4	1,7	1,4	6,4		20,8
Demand totals		801,5	469,4	199,4	57,4	194,4	57,0	207,5	41,9	8,2	252,9	8,3	265,5

Table 5.2

Input/output matrix (part 2/2)

		C (private consumption)						G (gov. consumption)	X (exports)				
		cBol	cBil	cEne	cVar	cTje	cTur	g	xEne	xVar	xSoe	xTje	xTur
Domestic production	Agriculture (lan)			0,1	4,7	1,2		0,5		21,9		0,0	
	Construction (byg)	3,7				0,7		7,4				33,6	
	Energy provision (ene)	3,9		49,2	0,0	0,0			8,8	0,0		0,0	
	Extraction (udv)	0,0			0,2	0,0			4,6	0,6		1,6	
	Housing (bol)	226,3			0,0	0,0							
	Manufacturing (fre)	1,8	0,1	0,0	54,2	1,7		1,1	0,0	424,0		22,1	
	Public (off)					60,3		543,2					
	Services, other (tje)	11,0	15,7	12,9	202,5	255,2		17,4	0,1	134,5	10,3	174,1	
	Shipping (soe)				0,0	2,2		0,2			220,4		
Imports	Energy provision (ene)			5,6	0,0				6,0				
	Extraction (udv)				0,0				0,0				
	Manufacturing (fre)	1,3	32,3	2,1	106,1	4,6		5,3		179,8			
	Services, other (tje)	0,0			3,6	15,2	23,6	0,2			6,1		
Demand totals		247,9	48,1	69,9	371,4	341,2	23,6	575,4	19,6	760,9	236,8	231,5	

from a sector s has the same price (before taxes) irrespective of who buys it.[49]

The after-tax price may vary depending on the buyer, as indirect taxes can vary across demand components. For example, households generally face higher indirect taxes on private consumption than firms on intermediate inputs. Demand prices (paid by the buyer) are given by:

$$P^{Y\tau}_{d,s,t} = \left(1 + \tau^{Y}_{d,s,t}\right) P^{Y}_{s,t}$$

$$P^{M\tau}_{d,s,t} = \left(1 + \tau^{M}_{d,s,t}\right) P^{M}_{s,t}$$

where $P^{Y}_{s,t}$ and $P^{M}_{s,t}$ are the (domestic and foreign) prices received by producers, which are the same irrespective of the identity of the buyer and the τ's denote taxes. In particular, the indirect tax rates for domestic production and imports are compositions of customs, net duties, and value added tax (VAT) rates:

$$\tau^{Y}_{d,s,t} = \left(1 + \tau^{DutyY}_{d,s,t} - s^{Y}_{d,s,t}\right)\left(1 + \tau^{VATy}_{d,s,t}\right)\left(1 + \tau^{Reg}_{d,s,t}\right) - 1$$

$$\tau^{M}_{d,s,t} = \left(1 + \tau^{Cus}_{d,s,t}\right)\left(1 + \tau^{DutyM}_{d,s,t} - s^{M}_{d,s,t}\right)\left(1 + \tau^{VATm}_{d,s,t}\right)\left(1 + \tau^{Reg}_{d,s,t}\right) - 1$$

where $\tau^{DutyY}_{d,s,t}$ and $\tau^{DutyM}_{d,s,t}$ are duty rates, $s^{Y}_{d,s,t}$ and $s^{M}_{d,s,t}$ are subsidy rates, $\tau^{VATy}_{d,s,t}$ and $\tau^{VATm}_{d,s,t}$ are the VAT rates, $\tau^{Reg}_{d,s,t}$ is the implicit registration duty for cars[50] and $\tau^{Cus}_{d,s,t}$ are the custom rates, all exogenous to the model and taken from the input/output data matrix in the National Accounts.[51]

5.2 Demand trees

Each demanded product, $D_{d,t} = [R_{r,t}, C_{c,t}, G_{g,t}, I_{i,t}, X_{x,t}]$, is a composition of different inputs produced in different sectors, s, either domestically (Y) or imported (M) .[52]

The explicit problem of the zero-profit retailer selling an aggregated final product to a specific market, $D_{d,t}$, is

[49] In the input/output matrix from the national accounts, it is possible to derive prices for the different IO cells. In these cells, the net price from each delivering sector will vary. We disregard this information to make the model more tractable. In the ADAM model, prices are not explicitly defined for the IO cells. Instead, they use constant IO coefficients in determining the aggregate price, which implicitly assumes all output from sector s has the same price irrespective of who buys it.
[50] The implicit registration duty, $\tau^{Reg}_{d,s,t}$, is only non-zero for private consumption of cars, investments in equipment and government consumption ($d = cCar, iM, g$).
[51] All duties, subsidies, customs, and VAT rates are allowed to vary across the demand and supply sectors. In the most dis-aggregate national accounts and in ADAM they are identical for all deliveries s. The variation on this dimension in MAKRO relative to ADAM is due to the fact that production sectors here aggregate a higher number of sub-sectors.
[52] Note that for intermediate inputs, $R_{r,t}$, r is an alias for s, i.e. r exists for all sectors.

$$
\begin{aligned}
\max_{Y_{d,st},M_{d,st}} \Pi^{retailer}_{d,t} &= P^{D}_{d,t} D_{d,t} - \textstyle\sum_{s} \left(P^{Y\tau}_{d,s,t} Y_{d,s,t} + P^{M\tau}_{d,s,t} M_{d,s,t} \right) \\
\text{s.t. } D_{d,t} &= \text{Leontief}\left(YM_{d,s1,t}, ..., YM_{d,S,t} \right) \\
\text{s.t. } YM_{d,s,t} &= CES\left(Y_{d,s,t}, M_{d,s,t} \right) \\
\text{s.t. } P^{D}_{d,t} D_{d,t} &= \textstyle\sum_{s} \left(P^{Y\tau}_{d,s,t} Y_{d,s,t} + P^{M\tau}_{d,s,t} M_{d,s,t} \right)
\end{aligned}
\tag{5.2}
$$

where $YM_{d,s,t}$ is a CES bundle over domestic and foreign inputs, $Y_{d,s,t}$ and $M_{d,s,t}$. The last equation in eq. (5.2) is the perfect competition condition that gives the final product price, $P^{D}_{d,t}$. Given the perfect competition condition this additive separable problem can be split into two cost minimization problems for a given level of demand $D_{d,t}$:

$$
\begin{aligned}
\min_{YM_{d,s,t}} & \textstyle\sum_{s} P^{YM}_{d,s,t} \cdot YM_{d,s,t} \\
\text{s.t. } D_{d,t} &= \text{Leontief}\left(YM_{d,s1,t}, ..., YM_{d,S,t} \right) \\
\text{s.t. } P^{D}_{d,t} D_{d,t} &= \textstyle\sum_{s} P^{YM}_{d,s,t} \cdot YM_{d,s,t}
\end{aligned}
$$

and

$$
\begin{aligned}
\min_{Y_{d,s,t},M_{d,s,t}} & P^{Y\tau}_{d,s,t} Y_{d,s,t} + P^{M\tau}_{d,s,t} M_{d,s,t} \\
\text{s.t. } YM_{d,s,t} &= CES\left(Y_{d,s,t}, M_{d,s,t} \right) \\
\text{s.t. } P^{YM}_{d,t} \cdot YM_{d,t} &= P^{Y\tau}_{d,s,t} Y_{d,s,t} + P^{M\tau}_{d,s,t} M_{d,s,t}
\end{aligned}
$$

The above structure does not apply for exports and inventory investments. For exports there are explicit demands for domestic exports and for imports for export - the two are not nested (see exports chapter). For inventory investments there is no aggregated demand for inventory investments, $I_{iL,t}$. Aggregate inventory investments is simply an accounting quantity imputed using chain indices. The real demand is for sectoral inventory investments, $I_{iL,s,t}$. Determination of inputs to exports and inventory investments have their own subsections below.

5.2.1 Aggregation of foreign and domestic inputs

Domestic and foreign inputs are aggregated using a CES function with a fixed elasticity of substitution $\eta \equiv \eta^{YM}_{d,s}$:

$$
YM_{d,s,t} = \left[\left(\mu^{Y}_{d,s,t} \right)^{\frac{1}{\eta}} \left(Y_{d,s,t} \right)^{\frac{\eta-1}{\eta}} + \left(\mu^{M}_{d,s,t} \right)^{\frac{1}{\eta}} \left(M_{d,s,t} \right)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}
$$

This is equivalent to solving an optimization problem with the accounting identity

$$P^{YM}_{d,s,t} \cdot YM_{d,s,t} = P^{Y\tau}_{d,s,t} Y_{d,s,t} + P^{M\tau}_{d,s,t} M_{d,s,t}$$

which generates the demand functions

$$Y_{d,s,t} = \mu^{y}_{d,s,t} \cdot YM_{d,s,t} \cdot \left(\frac{P^{Y\tau}_{d,s,t}}{P^{YM}_{d,s,t}} \right)^{-\eta^{YM}_{d,s}} \tag{5.3}$$

$$M_{d,s,t} = \mu^{m}_{d,s,t} \cdot YM_{d,s,t} \cdot \left(\frac{P^{M\tau}_{d,s,t}}{P^{YM}_{d,s,t}} \right)^{-\eta^{YM}_{d,s}} \tag{5.4}$$

where $P^{YM}_{d,s,t}$ is the corresponding zero-profit CES price aggregate of prices $\left(P^{Y\tau}_{d,s,t}, P^{M\tau}_{d,s,t}\right)$.

The key implication of using this CES structure to aggregate the quantities from foreign and domestic suppliers (instead of a price/quantity index construction often used in the data) is that it contains behavior which can be quantified through the elasticity parameter.

Reduced short-run elasticity We tweak the structure above (subsection 5.2.1) in order to impose lower short-run (relative to long run) elasticities between domestic and foreign inputs.

Specifically, we retain the zero profit condition along with the explicit quantity aggregator as before:

$$P^{YM}_{d,s,t} \cdot YM_{d,s,t} = P^{Y\tau}_{d,s,t} Y_{d,s,t} + P^{M\tau}_{d,s,t} M_{d,s,t}$$

$$YM_{d,s,t} = \left[\left(\mu^{Y}_{d,s,t}\right)^{\frac{1}{\eta}} \left(Y_{d,s,t}\right)^{\frac{\eta-1}{\eta}} + \left(\mu^{M}_{d,s,t}\right)^{\frac{1}{\eta}} \left(M_{d,s,t}\right)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

But, replace the ratio of the two demand equations

$$\frac{M_{d,s,t}}{Y_{d,s,t}} = \frac{\mu^{m}_{d,s,t}}{\mu^{y}_{d,s,t}} \cdot \left(\frac{P^{M\tau}_{d,s,t}}{P^{Y\tau}_{d,s,t}} \right)^{-\eta^{YM}_{d,s}}$$

with an ad-hoc generalization

$$\frac{M_{d,s,t}}{Y_{d,s,t}} = \frac{\mu^{m}_{d,s,t}}{\mu^{y}_{d,s,t}} \cdot \left(R^{YM}_{d,s,t}\right)^{-\eta^{YM}_{d,s}}$$

where the relative price object, $R^{YM}_{d,s,t}$, reacts sluggishly to changes in actual prices:

$$R^{YM}_{d,s,t} = \left(R^{YM}_{d,s,t-1}\right)^{\lambda_{d,s}} \left(\frac{P^{M\tau}_{d,s,t}}{P^{Y\tau}_{d,s,t}} \right)^{1-\lambda_{d,s}}$$

The resulting price $P^{YM}_{d,s,t}$ is no longer the standard CES price, but it remains the price consistent with zero profits. It is still a method of aggregation, except with added behavior.

5.2.2 Aggregation of quantities from different sectors

In this level we use a Leontief aggregator combining sectoral inputs into final goods in fixed proportions. For $d \in \{c, k, g, r\}$ we have

$$YM_{d,s,t} = \mu^{YM}_{d,s,t} D_{d,t}$$

In this expression $\mu^{YM}_{d,s,t} \in [0, 1]$ are the Leontief shares and $D_{d,t}$ are the final goods quantities. For example $\mu^{YM}_{c,s,t}$ is the fraction of total demand for consumption good c, $C_{c,t}$, that falls on goods produced or imported by sector s.

Note that for intermediate goods, $R_{r,t}$, r is an alias for s, i.e. r exists for all sectors. Finally, note that for investments we assume that all investment demand is made up of the same composition of sectoral inputs across sectors, implying that all sectors have identical investment prices. In national accounts, the price of investments may differ between sectors. Therefore, to ensure consistency with national accounts, we introduce a correction parameter in Appendix 5.4.1. The parameter has a linear effect on aggregate prices.

5.2.3 Exports

For exports there are explicit separate demand functions for domestically produced exports and imports that are re-exported. Domestically produced exports, $X^{Y}_{x,t}$, and imports for export, $X^{M}_{x,t}$, are each determined by Armington demand relations - see the exports chapter.

For each export demand, sectoral inputs are combined with a Leontief aggregator. The inputs are given by:

$$Y_{x,s,t} = \mu^{Xy}_{x,s,t} \cdot X^{Y}_{x,t}$$
$$M_{x,s,t} = \mu^{Xm}_{x,s,t} \cdot X^{M}_{x,t}$$

5.3 Aggregation

The production of each sector is the sum of deliveries to all demand components for both domestic production and imports:

$$Y_{s,t} = \sum_{d} Y_{d,s,t}$$
$$M_{s,t} = \sum_{d} M_{d,s,t}$$

At the sectoral level, there are model-defined equilibrium prices. However, at the aggregate level there are no model-defined prices. Thus, for totals to ensure consistency with National Accounts we use Paasche price indices and Laspeyres indices for the corresponding quantities.[53]

[53] We do the same for GDP and for aggregate gross value added. Note that housing and non-housing consumption

In particular, for a given variable, $Z = [C, I, G, R, X, M]$, we use the identity

$$P_t^Z Z_t = \sum_d P_{d,t}^Z Z_{d,t} \tag{5.5}$$

and the index relationship

$$P_{t-1}^Z Z_t = \sum_d P_{d,t-1}^Z Z_{d,t} \tag{5.6}$$

Together, (5.5) and (5.6) imply the price and quantity dynamic indices

$$Z_t = Z_{t-1} \frac{\sum_d P_{d,t-1}^Z Z_{d,t}}{\sum_d P_{d,t-1}^Z Z_{d,t-1}} \quad \text{and} \quad P_t^Z = P_{t-1}^Z \frac{\sum_d P_{d,t}^Z Z_{d,t}}{\sum_d P_{d,t-1}^Z Z_{d,t}}$$

Price indices are unnecessary when quantities are homogeneous. In such a case, we replace the index equation with the quantity sum and work with

$$P_t^Z Z_t = \sum_d P_{d,t}^Z Z_{d,t} \quad \text{and} \quad Z_t = \sum_d Z_{d,t}$$

5.4 Appendices

5.4.1 Details on investments

All firms buy the same composite investment goods. We assume that the contributions from supplying sectors, s, to a unit of a given type, k, of capital investment, $I_{k,t}$, are identical in all demand sectors. In addition, the contributions from domestic and foreign sources in the lowest level of the demand tree are also identical for all sectors. This implies that quantities are homogeneous and can be added across sectors to obtain the aggregate quantity demanded for an investment good. It also implies that the price of a unit of capital is the same across sectors, $P_{k,t}^I$.

However, in the national accounts investment prices for the same capital goods differ across sectors. To match this, we add a correction factor, $\lambda_{k,r,t}^{PI}$, and define sectoral prices as $P_{k,r,t}^I = \lambda_{k,r,t}^{PI} P_{k,t}^I$ where r is the sector demanding capital.[54] This price is then the relevant price for the optimal dynamic investment decision in each sector and is the price that enters the capital goods Euler equations, determining user costs. This factor $\lambda_{k,r,t}^{PI}$ enters after the two bottom CES constructions are decided. It affects aggregate prices linearly.

do not face this problem as they have model-defined aggregate prices through CES preferences. Investment quantities aggregate linearly so that we do not need a price index to calculate the price of total investment of a given type (structures or equipment). Using the price index approach does not affect the outcome in a significant way.

[54] It is easy to confuse the price from the perspective of the sector supplying inputs to the capital good $P_{k,s,t}^{Y\tau}$ with the price from the persepctive of the sector demanding the capital good $P_{k,r,t}^I$. This is why we use subscript r for the demanding sector here. Note that $P_{k,r,t}^I$ is a weighted average of $P_{k,s,t}^{Y\tau}$.

5.4.2 Data and calibration

Our sectoral aggregation and the resulting input/output matrix match the corresponding nominal aggregation from the Danish National Accounts. However, the data for the current version of MAKRO is based on the database from the ADAM-model. ADAM has 12 sectors, 8 private consumption groups, 1 government consumption group, 5 investments groups and 8 export groups. There is a direct mapping from ADAM's to MAKRO's consumption, investment and export groups. This mapping is as follows:

The production sector decomposition is almost a one-to-one mapping from ADAM to MAKRO. Agriculture (lan,a), construction (byg,b), extraction (udv,e) housing (bol,h), sea transport (soe,qs) are identical. Energy is decomposed in two (energy manufacturing, ne, and energy refinery, ng) in ADAM but joined in MAKRO (ene = ne+ng). Manufacturing is also decomposed (food nf and other nz) in ADAM and joined in MAKRO (fre = nf+nz). The private service sector in MAKRO is defined as all services, including public and financial services, and excluding all public services (offentlig forvaltning og service, o1 in ADAM). This yields the mapping for services (tje,qf+qz+o-o1), and for public services (off,o1).

The public sector is not a part of the official input/output system. We have initiated a project where MAKRO will receive input/output cells for the public sector's production. Until then private services are a residual and public services reflect the public sector. One disadvantage of having private services residual is that there is some public production in each sector and taking it all from services is only an approximation. Another is that there is no information on the input-structure from and to this definition of the public sector. This is solved by assuming that intermediate inputs to public production (off) are proportional to that of sector "o" in ADAM and by assuming that all deliveries from the public sector go to public sales, direct public investments, and public consumption. All public sales are assumed to go to private consumption of services, and all direct investments from the public sector are assumed to go to intellectual rights placed under investments in equipment - i.e. there are no exports from the public sector and no intermediate inputs from the public to the private sectors.

In ADAM it is assumed that, in every purchasing sector, investments in a given type of capital good contain the same input contributions from supplying sectors. National accounts data contains detailed information about the deliveries to investment types in the different sectors. MAKRO has the same assumption as ADAM, primarily to reduce the dimensionality of the input/output system. This does not change the number of markets that have to clear as the overall number of production sectors determines that. But it reduces the number of CES tree prices and quantities that must be computed. All sectors then have the same price index for investments in the input/output system.

Imports in ADAM are divided into product groups, whereas here, they are decomposed as the production sectors.[55] We include energy imports (from SITC Group 3) under imports from the

[55] Imports in ADAM are more disaggregated than in MAKRO. They are divided into food, coal, crude oil, other raw materials, other energy, cars, ships and aircraft, and other manufacturing. Imports of food inputs are substitutes for domestic food industry output (in MAKRO part of manufacturing). Imports of other raw materials are substitutes for manufacturing in ADAM (as in MAKRO), and manufacturing imports substitutes itself. In MAKRO other import groups do not substitute for domestic production. Ships and aircraft have no significant size, and cars are included primarily as input for car consumption, where the import share is so large that substitution is insignificant. However, in ADAM, it matters as they do not have substitution at the disaggregated IO cell level

foreign energy and extraction industries, other imports of goods under the foreign manufacturing industry, and service imports under the foreign service industry. All imports come from these 4 industries. Therefore, many $\mu^{M}_{d,s,t}$ parameters are zero. Imports of energy is exogenous so it has no endogenous substitution. Production in the extraction sector is exogenous so imports alone clear this market. This means that all substitution between import and domestic production is in relation to goods and services alone.

In the industry-disaggregated data from the national accounts IO tables, imports from construction, extraction, housing, and public production are extremely small. However, there are sizeable imports from foreign agriculture and shipping. This should, in principle, substitute for these domestic industries. However, it is not apparent how, as long as we rely on ADAM data. Therefore, we follow ADAM and let them substitute manufacturing and private services.

Before taxes, the bottom prices in MAKRO are market clearing prices identical for all buyers. This is not the case in the national accounts for our level of aggregation, and so the corresponding quantities are different in MAKRO and the national accounts. The aggregate quantities of sectoral imports and domestic production are scaled, so the quantities of aggregate deliveries from all sectors and import components to specific demand components are the same in MAKRO and the national accounts. The imputation of data using this assumption is made in the iodata_ADAM.gms file. All aggregates are calculated as Laspeyres quantity and Paasche price indices except on the assumed micro level. All share parameters in the IO equations are statically calibrated, so they correspond to MAKRO IO data.

5.4.3 Balancing share parameters

The share parameters, $\mu^{YM}_{d,s,t}$, $\mu^{Y}_{d,s,t}$ and $\mu^{M}_{d,s,t}$ are balanced endogenously to ensure consistency of the IO system in the case a user wants to shock e.g. the import share in a certain sector. For a shock to any share, other shares adjust such they always sum to 1. We construct the share parameters, $\mu^{YM}_{d,s,t}$, $\mu^{Y}_{d,s,t}$ and $\mu^{M}_{d,s,t}$ using the auxiliary exogenous variables $\mu^{YM_0}_{d,s,t}$, $\mu^{Y_0}_{d,s,t}$, $\mu^{M_0}_{d,s,t}$, $\lambda_{d,t}$, and $\lambda^{YM}_{d,s,t}$ as follows:[56]

$$\mu^{YM}_{d,s,t} = \lambda_{d,t}\frac{\mu^{YM_0}_{d,s,t}}{\sum_s \mu^{YM_0}_{d,s,t}}$$

$$\mu^{Y}_{d,s,t} = \lambda^{YM}_{d,s,t}\frac{\mu^{Y_0}_{d,s,t}}{\mu^{Y_0}_{d,s,t} + \mu^{M_0}_{d,s,t}}$$

$$\mu^{M}_{d,s,t} = \lambda^{YM}_{d,s,t}\frac{\mu^{M_0}_{d,s,t}}{\mu^{Y_0}_{d,s,t} + \mu^{M_0}_{d,s,t}}$$

for $d = r, c, g, i, x$.

In the calibration $\mu^{YM}_{d,s,t}$, $\mu^{Y}_{d,s,t}$ and $\mu^{M}_{d,s,t}$ are determined as usual. It is imposed that $\sum_s \mu^{YM_0}_{d,s,t} = 1$ and $\mu^{Y_0}_{d,s,t} + \mu^{M_0}_{d,s,t} = 1$. Then we have

but at the overall import group level.

[56] With this construction we can shock an individual deeper parameter indexed zero and the mechanics of the construction of the resulting parameters will spread the initial shock through all of them.

$$\mu^{YM}_{d,s,t} = \lambda_{d,t}\mu^{YM_0}_{d,s,t}$$

$$\mu^{Y}_{d,s,t} = \lambda^{YM}_{d,s,t}\mu^{Y_0}_{d,s,t}$$

$$\mu^{M}_{d,s,t} = \lambda^{YM}_{d,s,t}\mu^{M_0}_{d,s,t}$$

which implies $\lambda_{d,t} = \sum_s \mu^{YM}_{d,s,t}$ and $\lambda^{YM}_{d,s,t} = \mu^{Y}_{d,s,t} + \mu^{M}_{d,s,t}$.

Exceptions: Public direct investments and private consumption of public production

The exogenous share parameters, $\mu^{YM}_{d,s,t}$, $\mu^{Y}_{d,s,t}$ and $\mu^{M}_{d,s,t}$ are constructed using the auxiliary exogenous variables $\mu^{YM_0}_{d,s,t}$, $\mu^{Y_0}_{d,s,t}$, $\mu^{M_0}_{d,s,t}$, $\lambda_{d,t}$, and $\lambda^{YM}_{d,s,t}$. There are two exceptions to this structure, and they are the share parameter for deliveries from the public sector to private consumption, $\mu^{Y_0}_{c,s,t}$, and for deliveries from the public sector to investments, $\mu^{Y_0}_{i,s,t}$ where $s =$ public. The demand for public production for private consumption and direct investments is covered in the chapter on public production. As we model these demands directly, the variables do not follow the general demand for investment and private consumption inputs and the share parameters are adjusted endogenously to match.

6 Exports

We assume five export groups in MAKRO: energy, goods, shipping, tourism, and *other services*.[57] The Danish economy is significantly more open today than it was a few decades ago and we continue this trend by forecasting elements of the model governing the demand for exports.

Most exported goods are produced at home, but a small fraction consists of goods that are imported and immediately re-exported. These are goods in transit and are therefore treated separately. Any value added generated by the transit process is of course a part of exports, but, the value added from import-export business is accounted for in the export of domestically produced *other services*.

6.1 Export of domestically produced goods

6.1.1 Demand for the five export groups

The demand curve for each of the five export goods draws both on the Armington model - Armington (1969), Anderson (1979) - and on the Gravity equations from Anderson and Van Wincoop (2003). Using a single Armington type equation to describe the demand for exports of a particular good in a small open economy model carries a number of assumptions regarding aggregation of demand curves originating in different countries since aggregation is linear while the demand function is nonlinear. As the foreign demand for exports is exogenous to the model the specification we use is, while grounded in theory, empirically pragmatic.

For each export group, we specify a foreign demand function. Each export group,denoted with subscript x, is directly sourced from domestic production (superscript y), $X^{y}_{x,t}$:

$$X^{y}_{x,t} = \mu^{Xy}_{x,t} q^{x}_{t} q^{\text{Scale}}_{t} \left(R^{Xy}_{x,t}\right)^{-\eta^{X}_{x}}$$

As in the CES demand problem from Anderson (1979) exports relate to price ratios $R^{Xy}_{x,t}$ between the exported good price and its equivalent world price with a given elasticity η^{X}_{x}, and market size q^{x}_{t} approximates for the aggregate income of foreign consumers demanding the good. q^{Scale}_{t} is a scale parameter which measures the growth of domestic supply relative to world supply. In this way this equation is similar to any of the CES demand curves originating from our households or firms. A more detailed explanation of the variables are outlined below.

[57] Index set $x = \{xEne, xVar, xSoe, xTje, xTur\}$ where the prefix x stands for export.

6.1.2 Market size

The variable q_t^x is a dynamic construction which is constructed as an index of the import activity of Danish trade partners m_t^x:

$$q_t^x = (m_t^x)^{1-\psi^x} \left(q_{t-1}^x\right)^{\psi^x}$$

The parameter ψ^x controls rigidity in the transmission of increased economic activity among Danish trading partners to increased demand for Danish exports. The parameter is set so that it matches MAKRO's impulse responses to empirical models of shocks to foreign trade.

The import activity of Danish trade partners m_t^x is determined by the exogenously given measure of world output:

$$m_t^x = \mu_t^{m^x} Y^{world}$$

Where Y^{world} is the foreign gross value added and $\mu_t^{m^x}$ is a parameter which ties world output to foreign import expenditure.

6.1.3 Price ratio

The object $R_{x,t}^{Xy}$ is a relative price ratio construction which is constructed as the export price, $P_{x,t}^{Xy}$, relative to its export competing price, $P_{x,t}^{XF}$. This foreign price $P_{x,t}^{XF}$ is the "world price" in the respective export market and is described in the price setting chapter. The domestic export price $P_{x,t}^{Xy}$ reflects the way the export group x is sourced from the 9 domestic production sectors, and that composition is summarized by the factors $\mu_{x,s,t}^{YM}$.

$$p_{x,t}^{Xy} = \sum_s \mu_{x,s,t}^{YM} p_{x,s,t}^{YM}$$

We assume that there is no substitution between the sectors which an export group is sourced from - see the input/output chapter for details.

The object $R_{x,t}^{Xy}$ is constructed as follows:

$$R_{x,t}^{Xy} = \frac{P_{x,t}^{Xy}}{P_{x,t}^{XF}} - \Psi_t^{Xy} + \beta_{t+1} \frac{P_{t+1}^F}{P_t^F} \frac{X_{t+1}}{X_t} \Psi_{t+1}^{Xy}$$

where

$$\Psi_t^{Xy} \equiv \psi^{Xy} R_{x,t}^{Xy} \left[\frac{R_{x,t}^{Xy}}{R_{x,t-1}^{Xy}} - 1 \right] \left[\frac{R_{x,t}^{Xy}}{R_{x,t-1}^{Xy}} \right]$$

with the ψ^{Xy} parameter controlling *relative* price rigidity and β being a discount factor for foreign demand of exports. This construction brings a forward looking element into the demand for exports, and is derived by considering a foreign firm which buys from Denmark and then solves an optimal price setting problem when selling to its consumers in a market characterized by monopolistic competition and price rigidity.

We derive explicitly this equation in section 6.3. The usefulness of this construction is that it allows for changes in the prices $P_{x,t}^{Xy}$ or $P_{x,t}^{XF}$ to affect the quantity exported in a more dynamic and nuanced way than the effect obtained through the standard demand expression.

A low short-run export elasticity is a necessary component in generating substantial Keynesian effects. Again, ψ^{Xy} is one of the key parameters that we set in order to match MAKRO's impulse responses to a number of empirical models.

6.1.4 Export elasticity

The long-run export elasticities, η_x^X, are set to 5 based on estimations by Kronborg, A., Poulsen, K., & Kastrup, C. (2020). The export elasticity is a key parameter in MAKRO as it is the source of overall aggregate diminishing returns which allows the model to have a solution.

6.1.5 Scale

The scale variable q_t^{Scale} measures the growth of the domestic economy. It adds to the demand for exports an element of "supply generating its own demand". We use a dynamic construction

$$q_t^{\text{Scale}} = (1-\alpha)\sum_{i=0}^{n} \frac{Y_{t-i}^*/Y_t^{world}}{n+1} + \alpha \cdot q_{t-1}^{\text{Scale}}$$

where Y^*/Y^{world} is the ratio between structural output of the private sector and world output (gross value added), which q_t^{Scale} converges towards in an ARMA(1,5) process. α and n are parameters controlling the speed and path of adjustment (the ARMA process gives an s-shaped response of q_t^{Scale} to a permanent shift in Y^*).

The reason for Introducing q_t^{Scale} is the following: The demand curve that results from maximizing a CES objective function is a partial equilibrium object that cannot contain the income of the supply side. However, the solution to the trade model will contain it in some way, see Eaton and Kortum (2002), Melitz (2003). As we are not solving for world trade equilibrium we include this feature in a reduced form way. Technically what results is not a demand curve stricto sensu but rather a description of what drives the quantity exported.[58]

6.1.6 Energy exports

Energy exports are exogenous to the model and based on forecasts from the Danish Energy Agency.

[58] This solution is also suggested in the gravity model derived in Anderson and Van Wincoop (2003).

6.1.7 Tourism divided on consumption groups

The demand for domestically produced exports enters the input/output system directly, with the exception of the tourism export group which instead enters through the demand for consumption. Consumption including tourism is simply the sum of household consumption and tourist consumption. Both household and tourist consumption are separated into the various consumption groups. Determination of the household consumption is described in the household chapter. Tourist consumption, $X^{y}_{xTur,t}$, is divided on the different consumption groups, $C^{Tourist}_{c,t}$, with fixed weights, $\mu^{CTourist}_{c,t}$:

$$C^{Tourist}_{c,t} = \mu^{CTourist}_{c,t} X^{y}_{xTur,t}$$

The export price of tourism is a weighted average of the price of the consumption groups implicitly using the above weights $\mu^{CTourist}_{c,t}$.

Table 6.1 shows the input output matrix in MAKRO from a single year with emphasis on the demand for exports including tourism.

6.2 Imports for export

The demand for imports for export is modeled exactly as domestically produced exports above:

$$X^{m}_{x,t} = \mu^{Xm}_{x,t} q^{x}_{t} q^{\text{Scale}}_{t} \left(R^{Xm}_{x,t}\right)^{\eta^{X}_{x}}$$

except that the relative price $R^{Xm}_{x,t}$ is a static relation which only differs from 1 in the case of taxation of goods in transit. The foreign price competing with imports for export is simply the price of imports (see the chapter on price setting), and the relative price in the Armington equation for imports for export is

$$R^{Xm}_{x,t} = \frac{p^{Xm}_{x,t}}{\sum_s \mu^{m}_{x,s,t} P^{M}_{s,t}}$$

$$= \frac{\sum_s \mu^{m}_{x,s,t} P^{M\tau}_{d,s,t}}{\sum_s \mu^{m}_{x,s,t} P^{M}_{s,t}}$$

$$= \frac{\sum_s \mu^{m}_{x,s,t} \left(1 + \tau^{M}_{x,s,t}\right) P^{M}_{s,t}}{\sum_s \mu^{m}_{x,s,t} P^{M}_{s,t}}$$

While the numerator and denominator look very similar, the tax rate $\tau^{M}_{d,s,t}$ can be important for some (x, s) pairs and may vary over time.

Table 6.1
Input/output matrix

		R	C (private consumption)						G	I	X (exports)					
		rTot	cBol	cBil	cEne	cVar	cTje	cTur	g	iTot	xEne	xVar	xSoe	xTje	xTur	Supply totals
Domestic production	Agriculture (lan)	63,0			0,1	4,7	1,2		0,5	0,9		21,9		0,0		90,6
	Construction (byg)	75,6	3,7				0,7		7,4	228,2				33,6		349,3
	Energy provision (ene)	36,4	3,9		49,2	0,0	0,0			1,4	8,8	0,0		0,0		99,8
	Extraction (udv)	11,7	0,0			0,2	0,0			0,1	4,6	0,6		1,6		18,7
	Housing (bol)	0,0	226,3			0,0	0,0			0,1						226,4
	Manufacturing (fre)	230,7	1,8	0,1	0,0	54,2	1,7		1,1	58,0	0,0	424,0		22,1		793,8
	Public (off)						60,3		543,2	22,1						625,6
	Services, other (tje)	889,6	11,0	15,7	12,9	202,5	255,2		17,4	117,4	0,1	134,5	10,3	174,1		1840,6
	Shipping (soe)	19,8				0,0	2,2		0,2	0,2			220,4			242,9
Imports	Energy provision (ene)	42,1			5,6	0,0				0,2	6,0					53,9
	Extraction (udv)	10,6				0,0				0,1	0,0					10,5
	Manufacturing (fre)	267,5	1,3	32,3	2,1	106,1	4,6		5,3	73,0		179,8				672,1
	Services, other (tje)	389,5	0,0			3,6	15,2	23,6	0,2	27,1			6,1			465,4
Demand totals		2036,6	247,9	48,1	69,9	371,4	341,2	23,6	575,4	526,7	19,6	760,9	236,8	231,5		5489,5
Tourism export inside C					1,1	10,4	9,5								21,1	

2021 input/output table in current prices before taxes (bn. DKK). Columns representing demand for intermediate inputs or investments have been aggregated.

The red area shows domestically produced exports, whereas the blue area contains imports that are re-exported.

In each export column, the cells in red sum to $P^{Xy}_{x,t} X^{y}_{x,t}$, and the cells in blue sum to $P^{Xm}_{x,t} X^{m}_{x,t}$. Their total (in purple) is the total value of exports in export group x, $P^{X}_{x,t} X_{x,t}$.

The column for export of tourism is empty. Instead, export of tourism is part of the consumption cells shaded light red. The tourism-export values are shown in an auxiliary row below the input/output matrix, with the 3 consumption cells containing tourism exports summing to $P^{Xy}_{\text{xTur},t} X^{y}_{\text{xTur},t}$. Note that the import of tourism is completely unrelated and contains the consumption of Danish tourists abroad, whereas export of tourism are from foreign tourists in Denmark.

6.3 Derivation of the dynamic price ratio

Consider the demand for exports given by the Armington equation

$$X_t = \mu Q_t \left(\frac{\overline{P}_t^X}{P_t^F} \right)^{-\eta}$$

where $\overline{P}_t^X$ is the price of exports faced by foreign consumers and P_t^F is the price of competing foreign products. We now introduce intermediaries (located abroad) that buy Danish exports at price P_t^X and sell these exports to their consumers at the price $\overline{P}_t^X$. With free entry and no additional costs we have $\overline{P}_t^X = P_t^X$.

Assume now that intermediaries face monopolistic competition and have an adjustment cost of setting their price. The optimization problem becomes

$$\overline{P}_t^X = \arg\max_{p_{i,t}} \left[p_{i,t} - P_t^X\right] X_t \left[\frac{p_{i,t}}{\overline{P}_t^X}\right]^{-\eta} - \left[g_t + \beta_{t+1} g_{t+1}\right]$$

The first order condition of each intermediary is

$$X_t \left[\frac{p_{i,t}}{\overline{P}_t^X}\right]^{-\eta} \left(1 - \eta\left[p_{i,t} - P_t^X\right]\frac{1}{p_{i,t}}\right) - \left[\frac{\partial g_t}{\partial p_{i,t}} + \beta \frac{\partial g_{t+1}}{\partial p_{i,t}}\right] = 0$$

and imposing symmetric equilibrium $p_{i,t} = \overline{P}_t^X$ this writes

$$\overline{P}_t^X = \frac{\eta}{\eta - 1} P_t^X - \frac{1}{\eta - 1} \frac{\overline{P}_t^X}{X_t} \left[\frac{\partial g_t}{\partial p_{i,t}} + \beta \frac{\partial g_{t+1}}{\partial p_{i,t}}\right]$$

The expression shows that one can have zero (or close to zero) markups (the difference $\overline{P}_t^X - P_t^X$) without having perfect competition. On the other hand, as $\eta \to +\infty$ we do have that $\overline{P}_t^X = \rightarrow P_t^X$ (perfect competition implies zero markups).

There is also an adjustment cost in the expression above and is defined as the following:

$$g_t \equiv \frac{\hat{\psi}}{2} \left[\frac{p_{i,t}/P_t^F}{p_{i,t-1}/P_{t-1}^F} - 1\right]^2 \overline{P}_t^X X_t$$

Defining the price ratio $\overline{R}_t = \overline{P}_t^X / P_t^F$ and imposing symmetry yields

$$\overline{R}_t = \frac{\eta}{\eta - 1} \frac{P_t^X}{P_t^F} - \frac{1}{\eta - 1} \frac{\overline{R}_t}{X_t} \left[\frac{\partial g_t}{\partial p_{i,t}} + \beta \frac{\partial g_{t+1}}{\partial p_{i,t}}\right]$$

$$\frac{\partial g_t}{\partial p_{i,t}} = \hat{\psi} \left[\frac{\overline{R}_t}{\overline{R}_{t-1}} - 1\right] \left[\frac{\overline{R}_t}{\overline{R}_{t-1}}\right] X_t$$

and

$$\frac{\partial g_{t+1}}{\partial p_{i,t}} = -\frac{\overline{P}^X_{t+1}}{\overline{P}^X_t}\frac{\partial g_{t+1}}{\partial p_{i,t+1}}$$

Multiply by the factor $\frac{\eta-1}{\eta}$ to obtain the expression:

$$R_t \equiv \frac{\eta-1}{\eta}\overline{R}_t = \frac{P^X_t}{P^F_t} - \frac{1}{\eta-1}\frac{R_t}{X_t}\left[\frac{\partial g_t}{\partial p_{i,t}} + \beta\frac{\partial g_{t+1}}{\partial p_{i,t}}\right]$$

and define the auxiliary variable

$$\Psi^{Xy}_t \equiv \frac{1}{\eta-1}\frac{R_t}{X_t}\frac{\partial g_t}{\partial p_{i,t}} = \underbrace{\psi^{Xy}}_{\frac{\hat{\psi}}{\eta-1}} R_t\left[\frac{R_t}{R_{t-1}} - 1\right]\left[\frac{R_t}{R_{t-1}}\right]$$

we obtain the expression for the price ratio

$$R_t = \frac{P^X_t}{P^F_t} - \Psi^{Xy}_t + \beta\frac{P^F_{t+1}}{P^F_t}\frac{X_{t+1}}{X_t}\Psi^{Xy}_{t+1}$$

After having derived the dynamic relationship determining the price ratio we will comment some of the variables from above. Going back to the initial demand function we can see that using the R_t ratio is equivalent to redefining the scale parameter μ

$$X_t = \underbrace{\left[\frac{\eta}{\eta-1}\right]^{-\eta}\mu}_{\mu^{Xy}} Q_t\,(R_t)^{-\eta}$$

$$\mu^{Xy} = \left[\frac{\eta}{\eta-1}\right]^{-\eta}\mu$$

as well as redefining the adjustment cost parameter

$$\psi^{Xy} = \frac{\hat{\psi}}{\eta-1}$$

The value of η is not separately identifiable which implies we cannot judge how competitive the intermediary price setting market is. Irrespective, what matters is that the problem is well specified for any finite value of $\eta > 0$, and for any large value of η we have that $\mu^{Xy} \approx \mu$ and that, as it should, the price ratio approaches its spot value $R_t \rightarrow P^X_t/P^F_t$.

7 Labor market

The labor market contains heterogeneous households (unconstrained and "Hand-to-Mouth" type of households age 15 to 100) and firms. Households choose the amount of working hours and their labor market participation. Labor demand comes from firms posting their optimal amount of job vacancies, and a matching function brings vacancies and workers together. The model closes with bargaining between agents representing workers and firms, setting the market wage. The model aims to reproduce the level and behavior of Danish employment and wages.

A difficulty when formulating the labor market is that there is heterogeneity on both the supply and demand side - a life cycle with workers of different ages and multiple sectors on the firm side. We therefore build the model so that the household side is age-specific and the firm side is sector-specific, but the two dimensions are never present simultaneously. In addition, there are two types of households, the constrained/impatient and the unconstrained/patient, and we solve the model so that both types have the same labor market decisions. The following are the key assumptions we make:

- Firms cannot choose who they hire. Firms and workers meet randomly in the matching process, and once the meeting takes place, it is never optimal to send the worker away, irrespective of how old the worker might be.
- Optimal labor market participation and hours are age-specific but not sector specific. The worker cannot choose which sector she will be employed in and cannot voluntarily quit a job in one sector to join a different one.
- For computational reasons, we do not solve for the age distribution of workers inside each firm and instead impose that it is always the same for every firm. The data shows that the average age of workers is the same across firms of different sizes and for firms that are expanding or shrinking in terms of labor force size.

Finally, it is not the purpose of the model to unconditionally forecast long run trends in labor market participation or hours worked. Rather, these are exogenous inputs to the model's baseline based on detailed demographic projections including education, gender, and immigration status (Befolkningsregnskab).

7.1 Household problem

There is an exogenous number, $N_{a,t}$, of households of age a in period t. In this chapter, we interpret each household as containing a unit mass of agents who share the unemployment risk. We suppress the index within cohort heterogeneity, j, as we let labor supply be identical for all households of a given age. Households aged a at time t survive into the following age and period with probability $s_{a,t}$. When a household dies its entire unit mass of members dies with it.

Let $0 < q^e_{a,t} < 1$ be the fraction of the household unit mass of workers employed in period t. At the start of each period, agents in a household of age a can be in one of two labor market states: those who have kept their jobs from the previous period, $(1-\delta_{a,t})\, q^e_{a-1,t-1}$, and the remaining $1-(1-\delta_{a-1,t-1})\, q^e_{a-1,t-1}$ that start the period without a job. The object $\delta_{a,t}$ is an exogenous age-specific job destruction rate. After decisions are taken in period t and the market clears $q^e_{a,t}$ are employed. From the household point of view employment evolves according to

$$q^e_{a,t} = (1-\delta_{a,t})\, q^e_{a-1,t-1} + x_t q^s_{a,t} \tag{7.1}$$

where $q^s_{a,t}$ is search effort and x_t is the job finding probability. We now define a measure of labor force participation[59] as the sum of the employed and the unsuccessful search effort:

$$q^{lf}_{a,t} \equiv q^e_{a,t} + (1-x_t) q^s_{a,t} \tag{7.2}$$

Total employment, $n_t = n^e_t + n^f_t$, contains the employment of residents, $n^e_t = \sum_a q^e_{a,t} N_{a,t}$, plus a number of workers that are not residents, n^f_t (see section 7.2).

7.1.1 Household problem

We extend household utility to include disutility from hours worked, $h^e_{a,t}$, and from labor market participation, $q^{lf}_{a,t}$. We have then

$$U_{a,t}(C_{a,t}, D_{a,t}) - \underbrace{Z^h_{a,t}}_{\text{Control}} \underbrace{\rho^e_{a,t} q^e_{a,t}}_{\text{Scaling}} \underbrace{\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h} \left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{Hours}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}} + \underbrace{Z^{lf}_{a,t}}_{\text{Control}} \underbrace{\frac{\left(\lambda^{lf}_{a,t}\right)^{\eta^{lf}}}{1-\eta^{lf}} \left(1 - q^{lf}_{a,t}\right)^{1-\eta^{lf}}}_{\text{Participation}}$$

$$U_{a,t}(C_{a,t}, D_{a,t}) - \underbrace{Z^h_{a,t}}_{\text{Control}} \underbrace{\rho^e_{a,t} q^e_{a,t}}_{\text{Scaling}} \underbrace{\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h} \left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{Hours}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}} + \underbrace{Z^{lf}_{a,t}}_{\text{Control}} \underbrace{\frac{1}{1-\eta^{lf}} \left(1 - \lambda^{lf}_{a,t} q^{lf}_{a,t}\right)^{1-\eta^{lf}}}_{\text{Participation}}$$

The $Z_{a,t}$ controls are exogenous to the household and used to control for stationarity, and, in line with Galí, Smets, and Wouters (2012), to eliminate the marginal utility of consumption from the first order conditions. $\Xi^h_{a,t}$ is a control relating to hours which is exactly equal to the hours term, such that the participation choice is not affected by a second order effect relating to hours. This is similar to a subsistence consumption assumption, just for hours worked. The objects $\lambda^{lf}_{a,t}$ and $\lambda^h_{a,t}$ are calibration parameters. As job search and hours worked vary by age in the data the disutility parameters, $\lambda^{lf}_{a,t}$ and $\lambda^h_{a,t}$, vary by age to match these patterns. Note that the CRRA disutility of labor market participation binds labor market participation $q^{lf}_{a,t}$ below one for all age groups; and as the probability of finding a job is also in the unit interval, so is employment $q^e_{a,t}$.

Utility maximization is subject to the *budget constraint*

[59] We calibrate this measure to be identical to the concept of the *gross labor force* (brutto-arbejdsstyrke) in aggregate, but not by age.,

$$\Pi_{a,t} = \underbrace{(1-\tau_{a,t})\, w_t \rho^e_{a,t} h^e_{a,t}}_{\text{yearly wages after taxes}} \underbrace{q^e_{a,t}}_{\text{employment}} \tag{7.3}$$

$$+ (1-\tau_{a,t})\, b_{a,t} \underbrace{\left(q^{lf}_{a,t} - q^e_{a,t}\right)}_{\text{unemployment}} \tag{7.4}$$

$$+ (1-\tau_{a,t})\, b_{a,t} \underbrace{\left(1 - q^{lf}_{a,t}\right)}_{\text{out of the labor force}} \tag{7.5}$$

The household's after-tax wage depends on their effective tax rate $\tau_{a,t}$, the market wage w_t, age specific productivity $\rho^e_{a,t}$ and finally the number of hours worked $h^e_{a,t}$. $b_{a,t}$ is the unemployment compensation received by those that are unemployed or out of the labor force, and $\Pi_{a,t}$ summarizes all other budget constraint terms which do not appear in the problem at hand. As we do not make compensation contingent on search, the budget constraint simplifies to

$$\Pi_{a,t} = b_{a,t} + \left[(1-\tau_{a,t})\, w_t \rho^e_{a,t} h^e_{a,t} - b_{a,t}\right] q^e_{a,t} \tag{7.6}$$

which says that every household member earns $b_{a,t}$, and working members earn an additional wage premium $(1-\tau_{a,t})\left[w_t \rho^e_{a,t} h^e_{a,t} - b_{a,t}\right]$. As the tax rate depends on the household's labor market participation and their number of hours worked, we write the wage premium more generally as $\frac{\partial \Pi_{a,t}}{\partial q^e_{a,t}}$. From this wage premium we define $r^b_{a,t}$ as the net benefit replacement rate of a household on the extensive margin of the labor market, i.e. the income after taxes and subsidies received by a household that is not employed, divided by the income received by that household if they found a job.

$$1 - r^b_{a,t} \equiv \frac{\partial \Pi_{a,t} / \partial q^e_{a,t}}{(1-\tau_{a,t})\, w_t \rho^e_{a,t} h^e_{a,t}}$$

This rate determines the incentive to search for a job and participate in the labor market. The net replacement rate $r^b_{a,t}$ is exogenous to the model and based on calculations from the Ministry of Finance's models of rules and administrative data (Lovmodeller).

The effective tax rate, $\tau_{a,t}$, is likewise a complicated function of income and deductions, but crucially depends on the number of hours worked by the household. Rather than specifying this function here, we define the marginal income tax rate directly as:

$$1 - \tau^m_{a,t} \equiv \frac{\partial \Pi_{a,t} / \partial h^e_{a,t}}{w_t \rho^e_{a,t}}$$

The level of age specific marginal tax rates $\tau^m_{a,t}$ are also based on Ministry of Finance calculations, but move endogenously with shocks to tax rates.

We are now ready to maximize utility subject to the budget constraint and obtain the first-order conditions for participation and hours.

7.1.2 Optimal choice of hours

The first order condition for household $h^e_{a,t}$ is:

$$\left[1-\tau^m_{a,t}\right] U^c_{a,t} \frac{w_t \rho^e_{a,t}}{p^c_t}$$

$$U^c_{a,t}\left[1-\tau^m_{a,t}\right]\frac{w_t}{p^c_t} = Z^h_{a,t}\left(\lambda^h_{a,t} h^e_{a,t}\right)^{\eta^h} \tag{7.7}$$

where $Z^h_{a,t} = Z^c_{a,t} Z^\tau_{a,t} Z^w_t$. The term Z^w_t is used to control the impact of after-tax real wages. The term $Z^c_{a,t}$ is used to eliminate the wealth effect from this equation.[60] We let the Z terms be defined as

$$Z^w_t = \frac{w_t}{p^c_t}$$

$$Z^c_{a,t} = U^c_{a,t}$$

Eliminating the impact of real wages and marginal utility of consumption, the only remaining endogenous channel is from the marginal income tax rate $\tau^m_{a,t}$

$$h^e_{a,t} = \frac{1}{\lambda^h_{a,t}}\left[\frac{1-\tau^m_{a,t}}{Z^\tau_{a,t}}\right]^{\frac{1}{\eta^h}} \tag{7.8}$$

By default, this effect is also eliminated by setting $Z^\tau_{a,t} = \left[1-\tau^m_{a,t}\right]$, however model users can enable endogenous effects of shocks to marginal taxes on hours worked by exogenizing $Z^\tau_{a,t}$.[61] In that case, the elasticity of hours worked to the marginal tax rate is

$$\frac{\partial \ln h^e_{a,t}}{\partial \tau_{a,t}} = -\frac{1}{\eta^h\left[1-\tau^m_{a,t}\right]}$$

7.1.3 Optimal choice of labor market participation

Inserting the household law of motion of employment, (7.1), in the definition of $q^{lf}_{a,t}$ (7.2) we get an expression for $q^{lf}_{a,t}$ (labor force participation) as a function of the household's choice of employment rate:

$$q^{lf}_{a,t} = \frac{q^e_{a,t}}{x_t} - \left(\frac{1-x_t}{x_t}\right)\left(1-\delta_{a,t}\right) q^e_{a-1,t-1} \tag{7.9}$$

Using this and the first order condition for the $q^e_{a,t}$ (employment) yields the optimality condition:

[60] The standard is to assume $Z^c_{a,t}$ is a function of average marginal utility which in symmetric equilibrium equals the individual marginal utility. With the pooling assumptions within the unit mass of each household, and the assumption that all households are identical the average and the marginal are always identical but the symmetric equilibrium concept remains.

[61] We do not model the long run downward trend of the workweek. Z^c_t rules out the income effect (higher consumption implying lower marginal utility) and $Z^\tau_{a,t}$ rules out the substitution effect of higher taxes (funding the expanding welfare state) in the baseline. An trend in preferences $\lambda^h_{a,t}$ can be used to match an exogenously imposed trend in the baseline.

$$U_{a,t}(C_{a,t},D_{a,t}) - \underbrace{Z^h_{a,t}}_{\text{Control}}\underbrace{\rho^e_{a,t}q^e_{a,t}}_{\text{Scaling}}\underbrace{\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{Hours}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}} + \underbrace{Z^{lf}_{a,t}}_{\text{Control}}\underbrace{\frac{\left(\lambda^{lf}_{a,t}\right)^{\eta^{lf}}}{1-\eta^{lf}}\left(1-q^{lf}_{a,t}\right)^{1-\eta^{lf}}}_{\text{Participation}}$$

$$U_{a,t}(C_{a,t},D_{a,t}) - \underbrace{Z^h_{a,t}}_{\text{Control}}\underbrace{\rho^e_{a,t}q^e_{a,t}}_{\text{Scaling}}\underbrace{\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{Hours}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}} + \underbrace{Z^{lf}_{a,t}}_{\text{Control}}\underbrace{\frac{1}{1-\eta^{lf}}\left(1-\lambda^{lf}_{a,t}q^{lf}_{a,t}\right)^{1-\eta^{lf}}}_{\text{Participation}}$$

$$\begin{aligned} U_{a,t}(C_{a,t},D_{a,t}) - \underbrace{Z^h_{a,t}}_{\text{Control}}\underbrace{\rho^e_{a,t}q^e_{a,t}}_{\text{Scaling}}\underbrace{\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{Hours}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}} \\ + \underbrace{Z^{lf}_{a,t}}_{\text{Control}}\underbrace{\frac{\left(\lambda^{lf}_{a,t}\right)^{\eta^{lf}}}{1-\eta^{lf}}\left(1-\frac{q^e_{a,t}}{x_t}+\left(\frac{1-x_t}{x_t}\right)\left(1-\delta_{a,t}\right)q^e_{a-1,t-1}\right)^{1-\eta^{lf}}}_{\text{Participation}} \end{aligned}$$

$$\begin{aligned} U_{a,t}(C_{a,t},D_{a,t}) - \underbrace{Z^h_{a,t}}_{\text{Control}}\underbrace{\rho^e_{a,t}q^e_{a,t}}_{\text{Scaling}}\underbrace{\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{Hours}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}} \\ + \underbrace{Z^{lf}_{a,t}}_{\text{Control}}\underbrace{\frac{1}{1-\eta^{lf}}\left(1-\lambda^{lf}_{a,t}\frac{q^e_{a,t}}{x_t}+\lambda^{lf}_{a,t}\left(\frac{1-x_t}{x_t}\right)\left(1-\delta_{a,t}\right)q^e_{a-1,t-1}\right)^{1-\eta^{lf}}}_{\text{Participation}} \end{aligned}$$

$$\underbrace{Z^{lf}_{a,t}}_{\text{Control}}\underbrace{\left(\lambda^{lf}_{a,t}\right)^{\eta^{lf}}\left(1-\frac{q^e_{a,t}}{x_t}+\left(\frac{1-x_t}{x_t}\right)\left(1-\delta_{a,t}\right)q^e_{a-1,t-1}\right)^{-\eta^{lf}}}_{\text{Participation}} = x_t\left[\underbrace{Z^h_{a,t}\rho^e_{a,t}\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{disutility from hours worked}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}}\right]$$

$$\underbrace{Z^{lf}_{a,t}}_{\text{Control}}\lambda^{lf}_{a,t}\underbrace{\left(1-\lambda^{lf}_{a,t}\frac{q^e_{a,t}}{x_t}+\lambda^{lf}_{a,t}\left(\frac{1-x_t}{x_t}\right)\left(1-\delta_{a,t}\right)q^e_{a-1,t-1}\right)^{-\eta^{lf}}}_{\text{Participation}} = x_t\left[\underbrace{Z^h_{a,t}\rho^e_{a,t}\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{disutility from hours worked}} + \underbrace{\Xi^h_{a,t}}_{\text{Control}}\right]$$

$$\underbrace{Z^{lf}_{a,t}\left(\frac{\lambda^{lf}_{a,t}}{1-q^{lf}_{a,t}}\right)^{\eta^{lf}}}_{\text{marginal disutility of search}} = \underbrace{x_t}_{\text{probability of finding a job}} \cdot$$

$$\left\{\underbrace{\frac{U^c_{a,t}}{p^c_t}\left(1-r^b_{a,t}\right)\left(1-\tau_{a,t}\right)w_t\rho^e_{a,t}h^e_{a,t}}_{\text{wage premium}} - \left[\underbrace{Z^h_{a,t}\rho^e_{a,t}\frac{\left(\lambda^h_{a,t}\right)^{\eta^h}}{1+\eta^h}\left(h^e_{a,t}\right)^{1+\eta^h}}_{\text{disutility from hours worked}} + \underbrace{\Xi^h_{a,t}}_{\text{Control=disutility from hours worked}}\right]\right.$$

$$\left. + \underbrace{s_{a,t}\left(1-\delta_{a+1,t+1}\right)}_{\text{probability of surviving with a job}} \underbrace{\beta}_{\text{discount factor}} \underbrace{\left(\frac{1-x_{t+1}}{x_{t+1}}\right)Z^{lf}_{a+1,t+1}\left(\frac{\lambda^{lf}_{a+1,t+1}}{1-q^{lf}_{a+1,t+1}}\right)^{\eta^{lf}}}_{\text{saved disutility of search}}\right\} \tag{7.10}$$

Optimality trades off current against future marginal utility, where β is the relevant utility discount factor and $s_{a,t}$ is the survival rate. Extra engagement in the labor market today brings an immediate disutility of search, but with probability x_t (the job finding rate) leads to additional employment. In case of finding a job we earn a wage premium $\left(1-r^b_{a,t}\right)(...)$, but get disutility from hours worked. With probability $s_{a,t}\left(1-\delta_{a,t}\right)$ we are alive and still have the job the following period and save the disutility from having to search. Specifically, a retained job means we can lower $q^{lf}_{a+1,t+1}$ by $\left(\frac{1-x_{t+1}}{x_{t+1}}\right)$ and still get the same level of employment, as shown by the law of motion for q^{lf} (7.9).

We now define $Z^{lf}_{a,t} = \underbrace{Z^c_{a,t}Z^\tau_{a,t}Z^w_t}_{Z^h_{a,t}} Z^\rho_{a,t}Z^{r^b}_{a,t}$.

As before, $Z^c_{a,t}$ eliminates the effect of the marginal utility of consumption, Z^w_t eliminates the effect of real wages pr. productive hour, and $Z^\tau_{a,t}$ eliminates the effect of marginal taxes. In addition, we define

$$Z^\rho_{a,t} = \rho^e_{a,t}h^e_{a,t}$$

which eliminates the effect of age-specific productivity and hours worked.

Dividing through by $Z^{lf}_{a,t}$, the participation condition (7.10) can be re-written as

$$\underbrace{\left(\frac{\lambda^{lf}_{a,t}}{1-q^{lf}_{a,t}}\right)^{\eta^{lf}}}_{\text{marginal disutility of search}} = \underbrace{x_t}_{\text{probability of finding a job}} \cdot \left\{\underbrace{\frac{(1-\tau_{a,t})}{Z^{r^b}_{a,t} Z^{\tau}_{a,t}}\left(1-r^b_{a,t}\right)}_{\text{wage premium}} + \underbrace{s_{a,t}\left(1-\delta_{a+1,t+1}\right)}_{\text{probability of surviving with a job}} \underbrace{\beta\frac{Z^{lf}_{a+1,t+1}}{Z^{lf}_{a,t}}}_{\text{discount factor}} \underbrace{\left(\frac{1-x_{t+1}}{x_{t+1}}\right)\left(\frac{\lambda^{lf}_{a+1,t+1}}{1-q^{lf}_{a+1,t+1}}\right)^{\eta^{lf}}}_{\text{saved disutility of search}}\right\}$$

$$\underbrace{\left(\frac{\lambda^{lf}_{a,t}}{1-q^{lf}_{a,t}/\mathbf{\Lambda_{a,t}}}\right)^{\eta^{lf}}}_{\text{marginal disutility of search}} = \underbrace{x_t}_{\text{probability of finding a job}} \cdot \left\{\underbrace{\frac{(1-\tau_{a,t})}{Z^{r^b}_{a,t} Z^{\tau}_{a,t}}\left(1-r^b_{a,t}\right)}_{\text{wage premium}} + \underbrace{s_{a,t}\left(1-\delta_{a+1,t+1}\right)}_{\text{probability of surviving with a job}} \underbrace{\beta\frac{Z^{lf}_{a+1,t+1}}{Z^{lf}_{a,t}}}_{\text{discount factor}} \underbrace{\left(\frac{1-x_{t+1}}{x_{t+1}}\right)\left(\frac{\lambda^{lf}_{a+1,t+1}}{1-q^{lf}_{a+1,t+1}/\mathbf{\Lambda_{a+1,t+1}}}\right)^{\eta^{lf}}}_{\text{saved disutility of search}}\right\}$$

The elasticity of labor supply with respect to changes in the net benefit replacement rate is

$$\frac{\partial \ln q^{lf}_{a,t}}{\partial r^b_{a,t}} = -\frac{1}{\eta^{lf}} x_t \frac{1}{Z^{r^b}_{a,t}} \frac{(1-\tau_{a,t})}{Z^{\tau}_{a,t}} \left(\frac{\lambda^{lf}_{a,t}}{1-q^{lf}_{a,t}}\right)^{-\eta^{lf}} \left(\frac{1-q^{lf}_{a,t}}{q^{lf}_{a,t}}\right)$$

Finally, we define the last control variable $Z^{r^b}_{a,t}$ as

$$\frac{1}{Z^{r^b}_{a,t}} = \left[\frac{(1-\tau_{a,t})}{Z^{\tau}_{a,t}}\left(1-r^b_{a,t}\right)\right]^{\eta^{lf}/\eta^b - 1}$$

The parameter η^{lf} determines the effect of labor market conditions, as measured by the job finding rate x_t, on labor force participation, while η^b corresponds to the inverse elasticity of labor market participation wrt. to the net compensation rate.[62]

[62] The elasticity η^{lf} induces too strong a reaction of participation to changes in the finding rate late in life (after age 80). Rather than make the elasticity age dependent we choose to exogenize a fraction of the labor force at those ages. The resulting employment income is aggregated, and it is the appropriate average that enters the

7.2 Non-resident workers

In our model, total employment includes employed residents in Denmark and non-resident workers. Above, we outlined the decisions made by resident households. However, we assume that firms do not distinguish between residents and non-residents when they hire. Non-residents workers are cross-border agents that work in Denmark, but live abroad most or all of the year. These non-resident workers are not part of the resident population $N_{a,t}$ and they do not have any direct consumption in Denmark. Non-resident workers provide search input, $n^{s,f}_{a,t}$, into the matching function and generate employment $n^{f}_{a,t}$. They face the same job destruction rates and die and migrate at the same rate as the locals, and they stay in their jobs when they are not destroyed but do not demand local consumption or housing. However, in order to match the data, they have different productivity and work different hours than resident workers.

Non-resident workers have the same probability of finding a job as local job searchers, x_t, and the number of employed non-resident workers obeys the law of motion

$$n_t^f = (1 - \delta_t)\, n_{t-1}^f + x_t n_t^{s,f}$$

The total number of cross-border persons who are either employed or searching for a job in Denmark can be written as

$$N_t^f = (1 - \delta_t)\, n_{t-1}^f + n_t^{s,f}$$

We assume that this total is exogenous. The number of non-resident workers searching for a job is then endogenous and given by

$$n_t^{s,f} = N_t^f - (1 - \delta_t)\, n_{t-1}^f$$

E.g. when the job-finding rate is higher, more potential migrant workers find employment, reducing the search input of non-resident workers in the following period.

Since non-resident workers only enter the model through the firm and the matching function, their age decomposition is irrelevant, and only their aggregate contribution matters. Our default assumption is that the number of hours worked pr. non-resident worker is proportional to that of Danish households

$$h_t^f = \mu_t^{hf} \frac{\sum h_{a,t}^e n_{a,t}^e}{\sum n_{a,t}^e}$$

where μ_t^{hf} is calibrated to match the data.

7.3 Firms

What follows applies to all private sector firms in the model while the public sector is treated differently. There is a unit mass of identical firms in each (private) sector s. Employment in

budget constraint and affects the consumption decisions, which do not have these exogenous components.

the firm is given by the measure n_t which sums residents and non-resident workers.

Total workers in sector s, $n_{s,t}$, contribute with a total amount of productive hours $\rho_{s,t}h_{s,t}n_{s,t}$, where $\rho_{s,t}$ is the productivity factor and $h_{s,t}$ is the hours factor in the firm. We cover their link to the productivity and hours of the households in section (7.5.1) on aggregation.

Firms post vacancies and the economy wide matching function, m_t, dictates their success in filling them (see section 7.4 for details on the matching function). The matching process occurs in period t, after which employment is determined and production occurs. The firm cannot affect the hours worked by its employees and takes them as given. An effort/utilization choice by the firm is added to the model to help generate pro-cyclical value added per worker. Utilization is discussed in the production chapter.

7.3.1 The problem of the firm

Firms maximize profits and choose an optimal amount of labor input. When firms post vacancies, they hire workers. Although workers are "attached" to hours and productivity, the choice variable for the firm is n as the firm takes m as given. Current profits (as relevant for the optimal employment decision) are given by

$$\max_{n_{s,t}} \pi_t^s = \left(1 - \tau_{s,t}^c\right) \left\{p_{s,t}^0 Y_{s,t}\left(L_{s,t}\right) - w_t\left(1 + \tau_{s,t}^L\right)\rho_{s,t}h_{s,t}n_{s,t}\right\} \tag{7.11}$$

where $\tau_{s,t}^c$ are corporate taxes and $Y_{s,t}$ is gross-output as a function of labor input sold at price $p_{s,t}^0$. Wages paid by firms pr. employee, $w_t\left(1 + \tau_{s,t}^L\right)\rho_{s,t}h_{s,t}$, contain payroll taxes and subsidies $\tau_{s,t}^L$, the market wage w_t paid pr. productivity unit, sector specific productivity $\rho_{s,t}$ and hours pr. worker $h_{s,t}$. The object $n_{s,t}$ is the *stock* of employees as the employment decision of the firm is a dynamic one, just as the choice of its capital stock. Below we will derive the user cost of labor, $p_{s,t}^L \equiv p_{s,t}^0 \partial Y_{s,t}/\partial L_{s,t}$ which is evaluated at firm optimization price p^0. This optimization price is derived in the production chapter. The production input $u_{s,t}L_{s,t}$ is a broad concept of "labor". It is the total utilization $u_{s,t}$ of the effective labor input $L_{s,t}$ in productivity units. The effective labor input is net of hiring costs which are "paid" in foregone labor input.

The first-order condition for employment trades off the marginal costs against the marginal added profit for the marginal worker, and is given by

$$\left(1 - \tau_{s,t}^c\right) w_t\left(1 + \tau_{s,t}^L\right)\rho_{s,t}h_{s,t} = \left(1 - \tau_{s,t}^c\right) p_{s,t}^L \frac{\partial L_{s,t}}{\partial n_{s,t}} + \left(1 - \tau_{s,t+1}^c\right)\beta_{s,t+1}p_{s,t+1}^L \frac{\partial L_{s,t+1}}{\partial n_{s,t}} \tag{7.12}$$

where

$$p_{s,t}^0 \frac{\partial Y_{s,t}}{\partial L_{s,t}} = P_{s,t}^L = p_{s,t}^0 \frac{\partial Y_{s,t}}{\partial\left(u_{s,t}L_{s,t}\right)} u_{s,t}$$

In the following subsections, we detail the components of the firm's problem.

7.3.2 Wages and production input

We define L as the net amount of productive labor input that goes into the CES production function. It contains sector specific productivity and hours $\rho_{s,t}h_{s,t}$, and the endogenous adjustment for the fraction of employment used in the hiring process which we denote by χ (detailed in subsection 7.3.3 below):

$$L_{s,t} = \rho_{s,t}h_{s,t}\left(1-\chi_{s,t}\right)n_{s,t}$$

The cost of hiring is defined in terms of units of labor lost to production so that the total number of heads actually producing output is given by $(1-\chi)n$. Finally, the choice variable for the firm is the number of workers, so that the relevant derivatives are

$$\frac{\partial L_{s,t}}{\partial n_{s,t}} = \rho_{s,t}h_{s,t}\left(1-\frac{\partial\left(\chi_{s,t}n_{s,t}\right)}{\partial n_{s,t}}\right)$$

$$\frac{\partial L_{s,t+1}}{\partial n_{s,t}} = -\rho_{s,t+1}h_{s,t+1}\frac{\partial\left(\chi_{s,t+1}n_{s,t+1}\right)}{\partial n_{s,t}}$$

7.3.3 Vacancy posting costs

A unit mass of identical firms in sector s posts vacancies v and the law of motion for employment n is

$$n_{s,t} = \left(1-\delta_t\right)n_{s,t-1} + m_t v_{s,t}$$

The cost of posting vacancies is incurred in units of employment. We define an auxiliary endogenous variable χ such that $\chi_{s,t}n_{s,t}$ equals total vacancy posting costs:

$$\chi_{s,t}n_{s,t} = \kappa v_{s,t} + \frac{\gamma}{2}n_{s,t}\left[\frac{n_{s,t}}{n_{s,t-1}}/\frac{n_{s,t-1}}{\overline{n}_{s,t-2}}-1\right]^2 + \lambda m_t v_{s,t}$$

Vacancy costs are comprised of three terms, 1) a linear vacancy posting cost, 2) a quadratic labor adjustment cost, and 3) a matching cost which can be calibrated to match data on the cost of training new employees. Note that in the current version of MAKRO this component is set to zero.

The derivatives of total vacancy costs, wrt. to the firm's choice of employment, that go into the first-order condition of the firm are

$$\frac{\partial\left(\chi_{s,t}n_{s,t}\right)}{\partial n_{s,t}} = \frac{\kappa}{m_t} + \frac{\gamma}{2}\left[\frac{n_{s,t}}{n_{s,t-1}}/\frac{n_{s,t-1}}{\overline{n}_{s,t-2}}-1\right]^2 + \gamma\left[\frac{n_{s,t}}{n_{s,t-1}}/\frac{n_{s,t-1}}{\overline{n}_{s,t-2}}-1\right]\left[\frac{n_{s,t}}{n_{s,t-1}}/\frac{n_{s,t-1}}{\overline{n}_{s,t-2}}\right] + \lambda$$

$$\frac{\partial\left(\chi_{s,t}n_{s,t}\right)}{\partial n_{s,t-1}} = -\left(1-\delta_t\right)\left[\frac{\kappa}{m_t}+\lambda\right] - 2\gamma\frac{n_{s,t}}{n_{s,t-1}}\left[\frac{n_{s,t}}{n_{s,t-1}}/\frac{n_{s,t-1}}{\overline{n}_{s,t-2}}-1\right]\left[\frac{n_{s,t}}{n_{s,t-1}}/\frac{n_{s,t-1}}{\overline{n}_{s,t-2}}\right]$$

where we have used the law of motion for employment to eliminate the vacancies variable, i.e. using that

$$v_{s,t} = \frac{n_{s,t} - (1-\delta_t)\, n_{s,t-1}}{m_t}$$

7.3.4 Detailing the first-order condition

By rearranging the first-order condition (7.12) we get the expression

$$p^L_{s,t} = w_t \left(1+\tau^L_{s,t}\right) \rho_{s,t} h_{s,t} \left[\frac{\partial L_{s,t}}{\partial n_{s,t}} + \left(\frac{1-\tau^c_{s,t+1}}{1-\tau^c_{s,t}}\right) \hat{\beta}_{s,t+1} \frac{\partial L_{s,t+1}}{\partial n_{s,t}}\right]^{-1}$$

where[63]

$$\hat{\beta}_{s,t+1} \equiv \beta_{s,t+1} \frac{p^L_{s,t+1}}{p^L_{s,t}}$$

Inserting the derivatives of L we see clearly that in lieu of vacancy posting costs, the user cost of labor is simply the wage after taxes:

$$p^L_{s,t} = w_t \left(1+\tau^L_{s,t}\right) \left[1 - \underbrace{\frac{\partial \left(\chi_{s,t} n_{s,t}\right)}{\partial n_{s,t}} - \left(\frac{1-\tau^c_{s,t+1}}{1-\tau^c_{s,t}}\right) \hat{\beta}_{s,t+1} \frac{\rho_{s,t+1} h_{s,t+1}}{\rho_{s,t} h_{s,t}} \frac{\partial \left(\chi_{s,t+1} n_{s,t+1}\right)}{\partial n_{s,t}}}_{\text{hiring costs}}\right]^{-1}$$

or vice versa: firms post vacancies until the wage equals the marginal product of labor (user cost of labor) except for the wedge caused by hiring costs:

$$w_t = p^L_{s,t} \frac{1}{1+\tau^L_{s,t}} \left[1 - \underbrace{\frac{\partial \left(\chi_{s,t} n_{s,t}\right)}{\partial n_{s,t}} - \left(\frac{1-\tau^c_{s,t+1}}{1-\tau^c_{s,t}}\right) \hat{\beta}_{s,t+1} \frac{\rho_{s,t+1} h_{s,t+1}}{\rho_{s,t} h_{s,t}} \frac{\partial \left(\chi_{s,t+1} n_{s,t+1}\right)}{\partial n_{s,t}}}_{\text{hiring costs}}\right]$$

Note that sector specific hours and productivity disappear. This is consistent with the assumption that firms pay the same wage pr. productive hour of labor, i.e. the sector differences only reflect that the average worker in each sector provides a different number of productive hours of labor. Shifts in the sector-composition of employment therefore do not affect the total productivity units of labor available and does not directly affect total wages.

[63] In order to ensure that actual employment converges with our model of structural employment on any stable growth path, the default setting is to exogenize $\hat{\beta}_{s,t+1}$ such that it remains as it is in the baseline when we shock the model. The first-order condition retains forward-looking behavior since the derivatives of L contain endogenous variables.

7.3.5 Link to CES optimization

The dynamic first-order condition provides the input to the CES minimization problem used in solving the overall problem of the firm. The CES function is

$$p^{KEL}_{s,t} KEL_{s,t} \equiv p^{KEL}_{s,t} \left[\left(\mu^{KE}_{s,t}\right)^{\frac{1}{\eta}} \left(KE_{s,t}\right)^{\frac{\eta-1}{\eta}} + \left(\mu^{L}_{s,t}\right)^{\frac{1}{\eta}} \left(u^{L}_{s,t} L_{s,t}\right)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

On the budget side of the CES problem we have the total cost associated with all the labor actually used, L:

$$p^{KEL}_{s,t} KEL_{s,t} \equiv p^{L}_{s,t} L_{s,t} + p^{KE}_{s,t} KE_{s,t}$$

where the object $p^{L}_{s,t}$ is the user cost of L. In the CES optimization problem we take a derivative with respect to $L_{j,t}$ and this yields

$$u^{L}_{s,t} L_{s,t} = \mu^{L}_{s,t} KEL_{s,t} \left(\frac{p^{L}_{s,t}}{p^{KEL}_{s,t}} \frac{1}{u^{L}_{s,t}} \right)^{-\eta}$$

These equations are identical for all inputs.

7.4 The matching function

We use the matching function from den Haan, Ramey & Watson (2000) which is defined in the following way:

$$m\left(v_t, n^s_t\right) = \frac{n^s_t}{\left[\left(v_t\right)^{1/\eta^m} + \left(n^s_t\right)^{1/\eta^m} \right]^{\eta^m}}$$

The total number of searchers is the aggregate of households by age as well as non-resident job-searchers:

$$n^s_t = n^{s,f}_t + \sum_a q^{lf}_{a,t} N_{a,t}$$

and total vacancies is the sum of vacancies posted by each sector:

$$v_t = \sum_s v_{s,t}$$

The labor market equilibrium condition states that the number of job searchers finding a job must be equal to the number of vacancies filled and relates the vacancy matching rate m to the job finding rate x:

$$x_t n^s_t = m_t v_t$$

The matching function specification has a few important properties. It has constant returns to scale and can be rewritten as a function of labor market tightness $\frac{v_t}{n_t^s}$:

$$m\left(\frac{v_t}{n_t^s}\right) = \left[\left(\frac{v_t}{n_t^s}\right)^{1/\eta^m} + 1\right]^{-\eta^m}$$

It is also bounded between zero and one. I.e. if there are no other vacancies, a firm that posts a vacancy is sure to fill it. However, if there are no job-searchers left the probability of filling a vacancy goes to zero. More importantly, the job finding rate x is also bounded between zero and one.[64] Combined with the condition that labor force participation $q_{a,t}^{lf}$ is bounded, this ensures that both unemployment and employment rates are positive for all ages. Finally, the parameter η^m controls how sensitive households' and firms' probabilities of finding a job or filling a vacancy are to changes in labor market conditions. η^m is one of the key parameters that we set in order to match MAKRO's impulse responses to a number of empirical models.

7.5 Aggregation

7.5.1 Hours and productivity

Total labor hours match both the sum of sectors (demand) and the sum over households and non-resident workers (supply):

$$h_t n_t = \sum_s h_{s,t} n_{s,t} = \sum_a \left(h_{a,t}^e n_{a,t}^e\right) + h_t^f n_t^f$$

Sector-specific hours per worker differ across sectors because workers who sort into different sectors also differ in how many hours they work. We assume these differences reflect selection and move with workers when they change sector. Sector hours per worker are therefore given by

$$h_{s,t} = f_t^h \cdot h_{s,t}^0$$

where $h_{s,t}^0$ is a calibrated structural parameter and f_t^h is an endogenous balancing factor. Non-resident workers' hours per worker are proportional to the resident average (see section 7.2). In the forecast, f_t^h ensures that total hours on the sector (demand) side match total hours on the household and cross-border worker (supply) side, i.e.

$$\sum_s h_{s,t} n_{s,t} = h_t n_t$$

Average productivity, ρ, is defined from the identity

$$\rho_t h_t n_t = \sum_a \left(\rho_{a,t}^e h_{a,t}^e n_{a,t}^e\right) + \rho_t^f h_t^f n_t^f$$

[64] The finding rate as a function of labor market tightness is $x\left(\frac{v_t}{n_t^s}\right) = \left[\left(\frac{v_t}{n_t^s}\right)^{-1/\eta^m} + 1\right]^{-\eta^m}$

To ensure that the total productive units by sector also aggregates to the total, i.e.

$$\rho_t h_t n_t = \sum_s \rho_{s,t} h_{s,t} n_{s,t} \tag{7.13}$$

Equation (7.13) is ensured through endogenous balancing factors that shift sector productivity while preserving calibrated relative differences across sectors. We first separate labor productivity units between the public and private sectors

$$\begin{aligned}\rho_t h_t n_t &= \sum_s \rho_{s,t} h_{s,t} n_{s,t} \\ &= \sum_{s \neq \text{Pub}} \rho_{s,t} h_{s,t} n_{s,t} + \rho_{\text{Pub},t} h_{\text{Pub},t} n_{\text{Pub},t}\end{aligned}$$

with public sector productivity $\rho_{\text{Pub},t}$ exogenous.

This ensures that a change in public employment has a proportional effect on the public sector payroll, before general equilibrium effects, and that changes in employment between private sectors has no direct effect on the public payroll. Note that $\rho_{\text{Pub},t}$ has no effect on public production (see the chapter on public production), but $\rho_{\text{Pub},t} w_t$ is the hourly wage paid by the government. Sector specific productivity is calibrated to match differences in hourly wages across sectors, and only reflect differences in productivity insofar as differences in wages in the data reflect differences in productivity.

Given public sector productivity, the productivity in each private sector is given by

$$\rho_{s,t} = f_t^{\rho} \cdot \mu_{s,t}^{\rho} \cdot \rho_{\text{Priv},t} \mid s \neq \text{Pub}$$

where $\mu_{s,t}^{\rho}$ is a calibrated parameter which controls for relative productivity (hourly wages) between private sectors and f_t^{ρ} is an endogenous balancing factor. In the forecast, f_t^{ρ} ensures that productive units in the private sector aggregate consistently, i.e.

$$\rho_{\text{Priv},t} h_{\text{Priv},t} n_{\text{Priv},t} = \sum_{s \neq \text{Pub}} \rho_{s,t} h_{s,t} n_{s,t}$$

The interpretation is as follows: Differences in wages between sectors are entirely due to differences in productivity tied to the composition of workers in different sectors. Shifts in the sector-composition of employment therefore does not affect the total productivity units of labor available and does not directly affect total wages.

7.5.2 Employment law of motion, firm's point of view

Total employment n_t follows the law of motion

$$n_t = (1 - \delta_t) n_{t-1} + m_t v_t$$

Since we make the necessary assumption to ensure non-resident workers have the same law of motion as residents we can write

$$1-\delta_t = \frac{\sum_a (1-\delta_{a,t})\, n^e_{a-1,t-1} \frac{N_{a,t}}{N_{a-1,t-1}}}{\sum_a n^e_{a-1,t-1}}$$

where the aggregate destruction rate δ_t is now endogenous (although exogenous to the firm). Unlike the age-specific job destruction rate, the aggregate job destruction rate δ_t includes job destruction due to migration or death - both effects are captured by the change in cohort size $\frac{N_{a,t}}{N_{a-1,t-1}}$ (see Appendix 7.7.1).

Because the firm cannot choose whom it hires, it effectively always hires the average job searcher. Then, as we impose the same age distribution inside every firm irrespective of sector, all firms are the same in this respect and face the same job destruction rate. Since they do not control who they hire, they do not control the job destruction rate either (see Appendix 7.7.2).

7.6 Wage determination

We have derived the equations determining optimal search/participation (labor supply) and optimal vacancy posting (labor demand), but in both cases taking the wage level as given. To determine the wage level, we use an adapted Nash bargaining problem.

We assume a unique bargaining agent on both sides of the market which aggregate, on one side, preferences of all firms from all sectors, and, on the other, preferences of all workers of all ages. We also assume that these agents, which we can call unions, are "distant" from their firm and worker constituents and solve a simplified problem on their behalf. These assumptions allow for a degree of freedom in setting up the surpluses that enter the bargaining problem. One other important assumption is that although we consider a single bargaining problem that includes all agents, the intervening unions disregard the impact of the wage on equilibrium employment. Finally, we model wage rigidity directly in the bargaining problem through an adjustment cost function that extends the usual Nash bargaining surplus.

7.6.1 Bargaining

The solution to the bargaining problem results in a bargained wage w_t. Wage rigidity is introduced by adding a cost function to the standard Nash formulation. This cost function captures a utility loss of changing wages as perceived by the bargaining agents.[65] It also makes the problem dynamic.

Let S_t denote the surplus of the unions representing the workers and J_t that of the firms. These surpluses which enter the Nash bargaining function are given by the difference between the value of agreement and the value of disagreement for each party. Disagreement is an out of equilibrium event that never happens in model (we do not model labor disputes explicitly).

[65] Ljungqvist and Sargent, (2017) discuss standard formulations of the bargaining problem. The cost function draws on Kravik & Mimir (2019) and we use it also in the price setting problem.

In the case of the union representing workers the surplus is

$$S\left(w_t\right) = \underbrace{\left(1-\bar{\tau}_t\right)\rho_t h_t n_t}_{S'_t} w_t - o_t$$

where $\bar{\tau}_t$ is the weighted average of the marginal income tax rates[66]

$$\bar{\tau}_t = \frac{\sum_a \tau^m_{a,t}\rho^e_{a,t}h^e_{a,t}n^e_{a,t} + \bar{\tau}_t\rho^f_t h^f_t n^f_t}{\rho_t h_t n_t} = \frac{\sum_a \tau^m_{a,t}\rho^e_{a,t}h^e_{a,t}n^e_{a,t}}{\sum_a \rho^e_{a,t}h^e_{a,t}n^e_{a,t}}$$

and o_t is an outside option value which is exogenous to the agents[67].

On the firm side the surplus is the difference between the marginal product of labor and the wage paid. Payroll taxes, quadratic hiring costs, and effective hours pr. worker differ across private sectors, but the bargaining problem is solved by a single, economy-wide firm-side agent. To keep the aggregate problem tractable we make the following sector-aggregation assumption: the firm-side agent treats every sector-specific quantity entering its surplus as equal to its private-sector aggregate. The approximation is exact whenever those rates are uniform across sectors, but otherwise ignores cross-effects between some of the sources of sector variation.

Under this assumption the firm surplus collapses to a product of private-sector aggregates,

$$J\left(w_t\right) = \left[1-\tau^c_t\right]\rho_t h_t n_t\left(p^L_t - \left[1-\chi_t\right]\left[1+\tau^L_t\right]w_t\right)$$

where $\rho_t h_t n_t$ denotes aggregate productive labor input over the private sectors. The share χ_t measures the fraction of labor resources absorbed by vacancy posting and matching rather than entering production as effective labor $q^L_t = \left(1-\chi_t\right)\rho_t h_t n_t$ (see subsection 7.3.2).

It is useful to divide the firm surplus into two parts: 1) the value of the labor to the firms after taxes, J^+_t, and 2) net wages paid, $J'_t \cdot w_t$:

$$J\left(w_t\right) = \underbrace{\left[1-\tau^c_t\right]\rho_t h_t n_t p^L_t}_{J^+_t} - \underbrace{\left[1-\tau^c_t\right]\rho_t h_t n_t\left[1-\chi_t\right]\left[1+\tau^L_t\right]}_{-J'_t} w_t$$

The firm-side ratio entering the bargaining FOC simplifies to

$$\frac{J^+_t}{-J'_t} = \frac{p^L_t}{\left[1-\chi_t\right]\left[1+\tau^L_t\right]}.$$

Suppressing taxes and the vacancy wedge makes it clear that the heart of the firm's bargaining surplus is the wedge between the wage and the marginal product of labor:

66 We assume that non-resident workers pay the same taxes as average resident workers.

67 We model the outside option as a constant share of the sum of wages, weighed by the job finding probability:

$$o_t = \mu^o S'_t x_t w_t$$

The expression for the outside option can be interpreted as a share of the period and foregone wage $1-\mu^o$ wasted in negotiations, after which the job seeker can find a different job with probability x_t. Without better empirical foundations we are not ready to directly link unemployment compensation to the wage bargaining process, and we allow for the idea that searching for a different job is also an important outside option in decentralized negotiations.

$$J(w_t) \approx \underbrace{\rho_t h_t n_t}_{\text{scale}} \left(\underbrace{p_t^L}_{\text{MPL}} - \underbrace{w_t}_{\text{wage}} \right)$$

The vacancy wedge, $[1-\chi_t]$, exists because the cost of posting vacancies is a sunk cost in the bargaining situation. The firm's expected value of posting a vacancy is zero in equilibrium, but the value of a vacancy that has turned into a match is positive. The value of a match is often called the matching surplus, and it is this surplus which is bargained over and divided into a bargaining surplus for the worker and a bargaining surplus for the firm.

The bargaining surpluses are mostly static and wage dynamics primarily enter the bargaining problem through an adjustment cost function. We define the bargaining loss function as

$$L_t = (S_t)^{1-\phi_t} (J_t)^{\phi_t} - G_t \tag{7.14}$$

where $G(\cdot)$ is a wage adjustment cost and ϕ_t is the Nash bargaining power parameter. We now specify the adjustment cost function $G_t \equiv g_t + \beta_{t+1} g_{t+1}$ where

$$g_t = \frac{\psi^\omega}{2} \left[\frac{w_t}{w_{t-1}} / \frac{w_{t-1}}{\overline{w}_{t-2}} - 1 \right]^2$$

and where $\overline{w}_{t-2}$ is an exogenous object which in equilibrium satisfies $w_{t-2} = \overline{w}_{t-2}$.

The bargaining problem then consists of maximizing the loss function (7.14) with respect to w_t, yielding

$$w_t = \phi_t \frac{o_t}{S_t'} - [1-\phi_t] \frac{J_t^+}{J_t'} + \underbrace{\left[\frac{(S_t)^{\phi_t} (J_t)^{1-\phi_t}}{S_t' J_t'} \frac{1}{w_t} \right]}_{\approx \text{constant}} \left[\frac{\partial g_t}{\partial w_t} w_t - 2\beta_{t+1} \frac{\partial g_{t+1}}{\partial w_{t+1}} w_{t+1} \right] \tag{7.15}$$

For computational efficiency we replace the term $\left[\frac{(S_t)^{\phi_t} (J_t)^{1-\phi_t}}{S_t' J_t'} \frac{1}{w_t} \right]$ with a constant as there is no discernible difference between IRFs from holding it constant vis-a-vis letting it be endogenous. Note that the wage rigidity term is zero in any steady state.

Inserting the firm-side ratio derived above together with the outside option of the worker representatives, $o_t = \mu^o x_t S_t' w_t$, and solving for w_t in the bargaining FOC yields the bargained wage

$$w_t = \left[\frac{1-\phi_t}{1-\phi_t \mu^o x_t} \right] \frac{p_t^L}{[1-\chi_t]\left[1+\tau_t^L\right]} + \{\text{wage rigidity}\}.$$

7.7 Appendices

7.7.1 Aggregating household employment with migration

Population flows obey

$$N_{a,t} = s_{a-1,t-1}N_{a-1,t-1} + I_{a,t} - E_{a,t}$$

where $I_{a,t}$ are immigrants and $E_{a,t}$ are emigrants. Households making the choice described above are those surviving from the previous period. Those that survive from on period to another and stay in the country are denoted as; $N_{a-1,t-1}s_{a-1,t-1} - E_{a,t}$. Emigrants $E_{a,t}$ are identical to other residents except that they leave the country. Employment for a household of age a evolves according to the accumulation-equation

$$q^e_{a,t} = (1-\delta_{a,t})\, q^e_{a-1,t-1} + x_t q^s_{a,t}$$

Immigrants $I_{a,t}$ come into the country and we assume they obtain the same employment $q^e_{a,t}$ as residents. We also assume that immigrants have the same search behavior as surviving households. However, they do not have an employment history in the country. This implies that some immigrants enter with jobs, so they do not have to search. This accounts for the employment quantity $q^I_{a,t}I_{a,t}$. We have then

$$q^e_{a,t} = x_t q^s_{a,t} + q^I_{a,t}$$

This sums to

$$\underbrace{N_{a,t}q^e_{a,t}}_{n^e_{a,t}} = (1-\delta_{a,t})\left(s_{a-1,t-1} - \frac{E_{a,t}}{N_{a-1,t-1}}\right) q^e_{a-1,t-1}N_{a-1,t-1}$$
$$+ x_t \underbrace{q^s_{a,t}\left(s_{a-1,t-1}N_{a-1,t-1} + I_{a,t} - E_{a,t}\right)}_{n^s_{a,t}\equiv N_{a,t}q^s_{a,t}} + q^I_{a,t}I_{a,t}$$

Now assume that

$$q^I_{a,t} \equiv (1-\delta_{a,t})\, q^e_{a-1,t-1}$$

so that

$$\begin{aligned} n^e_{a,t} &= (1-\delta_{a,t})\left(s_{a-1,t-1} - \frac{E_{a,t}}{N_{a-1,t-1}} + \frac{I_{a,t}}{N_{a-1,t-1}}\right) n^e_{a-1,t-1} + x_t n^s_{a,t} \\ &= (1-\delta_{a,t})\frac{N_{a,t}}{N_{a-1,t-1}} n^e_{a-1,t-1} + x_t n^s_{a,t} \end{aligned}$$

Given this construction, we do not have to know the number of immigrants and emigrants. All we need to know is the total population.

7.7.2 Imposing the same age distribution on every firm

The result of the household's first order condition for participation/search is that employment will vary by age. However, we do not want to add the sectoral dimension to the disaggregated employment variable. In order to do this, when we solve the problem of the firm we do not solve endogenously for the age distribution of workers inside the different firms/sectors indexed by j. Instead we impose exogenously that this distribution is the same across all firms in the economy.[68] We also choose not to allow for differences in the job destruction rate across sectors arising from other factors. Evidence from administrative data on wage earners indicates that the average age of the labor force is independent of firm size and also uncorrelated with whether firms are reducing or expanding their employment.[69]

We force the same distribution using the relationship

$$n_{a,s,t} \equiv \frac{n_{s,t}}{n_t} n_{a,t}$$

Given our assumptions, we never have to use the bigger object $n_{a,s,t}$, since on the production side the age distribution does not matter and so we only care about total employment inside the firm.

7.7.3 Different average wages across sectors

Although in our model both labor supply and demand are anonymous, resulting in all firms hiring the same average worker looking for a job, and employing the same average employed worker in the economy, we observe in the data that average wages differ across sectors. It is possible that this reflects the heterogeneity of workers employed in different sectors, a feature which is ruled out in our model. In order to match the data on both employment and average wage acoss sectors we need a reduced form mechanism that will allow us to do so without breaking the two sided anonymity of the labor market.

The mechanism described here attaches different productivities to workers working in different sectors, while the workers themselves are identical wherever they happen to work. A three sector example helps illustrate it. We first impose the identifying constraint which attaches a relative sectoral productivity factor ρ_t^i to sectoral employment, while keeping the total constant:

$$\rho_{1,t}^w n_t^1 + \rho_{2,t}^w n_t^2 + \rho_{3,t}^w n_t^3 = \sum_i n_t^i = n_t$$

Given this constraint, calculate the average wage per sector in the data and compute the ratios:

$$\frac{\rho_{1,t}^w}{\rho_{3,t}^w} = \frac{w_t^1}{w_t^3} = w_t^{13}, \quad \frac{\rho_{2,t}^w}{\rho_{3,t}^w} = \frac{w_t^2}{w_t^3} = w_t^{23}$$

[68] Different sectors will move differently over the cycle. And age specific population does not move evenly over time which implies neither will the labor force. All firms from all sectors hire the "average job searcher" from the currently available pool in the economy. As firms from different sectors hire different amounts over time the age composition of labor inside firms across sectors will differ, while it is the same in all firms within a sector. Keeping track of the age distribution within each firm/sector greatly increases the dimensionality of the model.
[69] Job Destruction and Job Finding Rates by Age in Danmark. Dream Working Paper, May 5, 2021.

Note: the objects w_t^{ij} are data for the period where data exists, and are forecasts for the subsequent periods. They are an exogenous input into the model. This allows for the endogenous calculation of $\rho_{3,t}^w$:

$$\rho_{3,t}^w = \frac{n_t}{w_t^{13} n_t^1 + w_t^{23} n_t^2 + n_t^3}$$

and of course of the other two as well. During data years we use observed average wages and employment, and in the forecasting years we use a forecast of relative average wages to calculate the $\rho_{j,t}^w$. This mechanism preserves the search model. It is consistent with the randomness of matching. Which means the household problem is unaffected because of the initial identifying constraint. And it can be interpreted as a proxy for heterogeneity.

7.7.4 Labor market variables

Table 7.1
Households and non-resident workers

Documentation	Code	Definition
$q^e_{a,t} \cdot N_{a,t}$	nLHh[a,t]	Employment
$n^e_{a,t}$	nLHh[a,t]	Employment
$q^{lf}_{a,t} \cdot N_{a,t}$	nSoegBaseHh[a,t]	Sum of job seeking and employed
η^{lf}	eDeltag	Parameter controlling the elasticity of labor force participation from changes in the job finding rate
η^h	eh	Inverse labor supply elasticity - intensive margin
$s_{a,t}$	rOverlev[a,t]	Survival rate
$\delta_{a,t}$	rSeparation[a,t]	Job separation rate
δ_t	rSeparation[aTot,t]	Job separation rate, total over cohorts
x_t	rJobFinding[t]	Share of job seekers who get a job
$\lambda^{lf}_{a,t}$	uDeltag[a,t]	Preference parameter for labor market participation
$\lambda^h_{a,t}$	uh[a,t]	Preference parameter for hours
$\rho^e_{a,t}$	qProdHh[a,t]	Age dependent productivity wrt. education and composition effects
$\tau^m_{a,t}$	mtInd[a,t]	Marginal income tax rate
$h^e_{a,t}$	hLHh[a,t]	Age dependent hours worked
$h^f_t \cdot n^f_t$	hLxDK[t]	Total work hours for non-resident workers
$r^b_{a,t}$	mrKomp[a,t]	Marginal net benefit replacement rate
$q^s_{a,t} \cdot N_{a,t}$	nSoegHh[a,t]	Household job seekers
n^f_t	nLxDK[t]	Non-resident workers
$n^{s,f}_t$	nSoegxDK[t]	Job seeking potential non-resident workers
N^f_t	nSoegBasexDK[t]	Sum of non-resident workers and job seeking potential non-resident workers.

Table 7.2
Firms

Documentation	Code	Definition
$n_{s,t}$	nL[s,t]	Employment by sector
$\rho_{s,t}$	qProd[s,t]	Productivity of labor by sector
$v_{s,t}$	nOpslag[s,t]	Vacancies by sector
$\hat{\beta}_{s,t}$	fDiskpL	Exogenous discount rate on demand for labor
κ	uOpslagOmk	Linear job posting cost
γ	uMatchOmkSqr	Quadratic job posting cost
λ	uMatchOmk	Linear job match cost
m_t	rMatch[t]	Share of job postings resulting in a match
$\chi_{s,t}$	rOpslagOmk[s,t]	Share of worker inputs used for hiring costs
$h_{s,t}$	hL2nL[s,t]	Sector dependent work hours per employee
$h_{s,t} \cdot n_{s,t}$	hL[s,t]	Total hours worked by sector
h_t^f	uhLxDK[t]	Scale parameter linking non-resident hours per worker to the resident average
$\frac{\partial(\chi_{s,t} n_{s,t})}{\partial n_{s,t}}$	dOpslagOmk2dnL[s,t]	Derivative of hiring costs wrt. number of employees
$\frac{\partial(\chi_{s,t} n_{s,t})}{\partial n_{s,t-1}}$	dOpslagOmk2dnLLag[s,t]	Derivative of hiring costs wrt. lagged number of employees
L_t	qL[t]	Effective labor
$\frac{\partial L_{s,t}}{\partial n_{s,t}}$	dqL2dnL[s,t]	Derivative of effective labor L wrt. number of employees
$\frac{\partial L_{s,t}}{\partial n_{s,t-1}}$	dqLd2nLlag[s,t]	Derivative of effective labor L wrt. lagged number of employees
$p_{s,t}^L$	pL[s,t]	User cost for effective labor in production function
$\tau_{s,t}^L$	tL[s,t]	Rate of net payroll taxes and subsidies, self-employed included
$\tau_{s,t}^c$	mtVirk[s,t]	Corporate marginal income tax by sector
$\beta_{s,t}$	fVirkDisk[s,t]	Firms' discount rate
$u_{s,t}$	rLUdn[s,t]	Capacity utilization of labor

Table 7.3
Matching and Wage Bargaining

Documentation	Code	Definition
η^m	eMatching	Exponent in matching function
w_t	pW[t]	Wage per productive unit of labor
$w_t \cdot \rho_t$	vhW[t]	Average hourly wage
ϕ_t	rLoenNash[t]	Nash bargaining weight
S'_t	dFF2dLoen[t]	Derivative of the union's value function in wage bargaining wrt. the wage
o_t	vFFOutsideOption[t]	Outside option value of the union in wage bargaining

8 Structural employment and output

MAKRO calculates structural employment endogenously based on a modified version of the labor market framework. Most importantly, we remove nominal frictions stemming from the rigidity in wage bargaining and remove the real rigidity stemming from matching frictions. The assumptions behind structural employment are described in more detail in Section 8.1.

From the estimates of structural employment, the model also calculates structural output, specifically structural gross value added. We define structural output as the output that would be produced given structural employment, but conditional on the actual capital stock. To compute this measure, the model employs an aggregate production function that takes actual capital and structural employment as inputs. This is explained in more detail in Section 8.2.

The estimates of structural employment and output are essential for calculating the structural budget balance, which considers the effects of the business cycle on government revenues and expenditures. The Ministry of Finance relies on the gaps between actual and structural levels of employment and output, which can be forecast using MAKRO.

Both the output gap and employment gap are critical macroeconomic indicators that are typically measured empirically. The structural model presented in this chapter aligns with the Ministry of Finance's estimation approach[70], allowing the model to be calibrated to match the estimated gaps. Therefore, the model can be utilized to close these gaps endogenously or provide conditional forecasts of the effects of shocks or policy interventions.

It is important to note that MAKRO does not unconditionally forecast the structural labor force or hours worked. Instead, these variables are treated as exogenous inputs based on detailed demographic projections that account for factors such as education, gender, and immigration status (Befolkningsregnskab). Therefore, the model's estimates of structural employment and output are conditional on the assumptions underlying the demographic projections.

8.1 Structural employment

Structural employment is primarily determined by the structural supply of labor - the labor force, which is an exogenous input in the model's baseline. Structural employment in MAKRO is caused by real frictions in matching workers and firms. These frictions are structural, but they are also a source of short run rigidity. To move from actual employment to a structural measure, we need to remove the short run rigidity effect of matching frictions while maintaining their effect on structural unemployment. We achieve this by letting structural employment reach a quasi steady state in each period. Employment levels per se cannot be constant with changing demographics, but rate measures such as employment relative to population and unemployment rates can.

[70] "Finansministeriets beregning af gab og strukturelle niveauer" (Finansministeriet, November 2020).

The labor market in MAKRO is constructed so that long run employment is independent of wages and prices.

8.1.1 Law of motion of structural employment

The law of motion of actual employment is

$$n^e_{a,t} = (1-\delta_a)\, n^e_{a-1,t-1} \frac{N_{a,t}}{N_{a-1,t-1}} + x_t n^s_{a,t}$$
$$= q^e_{a,t} N_{a,t} = (1-\delta_a)\, q^e_{a-1,t-1} N_{a,t} + x_t n^s_{a,t}$$

where $q^e_{a,t}$ is the employment rate, δ_a is the job separation rate, x_t is the finding rate and $n^s_{a,t}$ is cohort total search effort.

To get to structural employment we modify this law of motion. First, we replace the job finding rate and search effort with their structural counterparts x^*_t and $n^{s*}_{a,t}$. Second, we replace the lagged employment rate $q_{a-1,t-1}$ with the the structural employment rate in the current period $q^*_{a-1,t}$ while maintaining the life cycle aspect of employment by retaining the lag on the age index:

$$n^{e*}_{a,t} = (1-\delta_a)\, q^{e*}_{a-1,t} N_{a,t} + x^*_t n^{s*}_{a,t}$$
$$= (1-\delta_a)\, n^{e*}_{a-1,t} \frac{N_{a,t}}{N_{a-1,t}} + x^*_t n^{s*}_{a,t}$$

We can write the aggregate law of motion of structural employment as

$$n^{e*}_t = (1-\delta^*_t)\, n^{e*}_{t-1} + x^*_t n^{s*}_t$$

where the aggregate job structural destruction rate is

$$(1-\delta^*_t) = \frac{\sum_a (1-\delta_a)\, q^{e*}_{a-1,t} N_{a,t}}{\sum_a q^{e*}_{a-1,t-1} N_{a-1,t-1}}$$

8.1.2 Non-resident workers

The law of motion of non-resident workers is modified as above to remove the short run rigidity effect of matching frictions (by letting employment reach a quasi steady state in each period):

$$n^{f*}_t = (1-\delta^*_t)\, n^{f*}_t + x^*_t n^{s,f*}_t$$

Both the actual and structural labor supply of non-resident workers is based on the same exogenous input - the total number of cross-border persons who are either employed or searching for a job in Denmark, N_t^f.

Based on this gross labor supply, the structural job-search of non-residents is

$$n_t^{s,f*} = N_t^f - (1 - \delta_t^*) \, n_t^{f*}$$

8.1.3 Structural choice of labor market participation

To calculate a structural level of households' labor market participation, $q_{a,t}^{lf*} N_{a,t} = n_{a,t}^{lf*} = n_{a,t}^{e*} + (1 - x_t^*) n_{a,t}^{s*}$, we use the same optimality condition as in the actual labor market, except for replacing the job finding probability and hours worked with their structural counterparts:

$$\underbrace{\left(\frac{\lambda_{a,t}^{lf}}{1 - q_{a,t}^{lf*}}\right)^{\eta^{lf}}}_{\text{marginal disutility of search}} = \underbrace{x_t^*}_{\text{probability of finding a job}} \cdot \left\{ \underbrace{\frac{(1 - \tau_{a,t})}{Z_{a,t}^{r^b} Z_{a,t}^{\tau}} \left(1 - r_{a,t}^b\right)}_{\text{wage premium}} - \underbrace{\frac{1}{Z_{a,t}^{r^b}} \frac{\left(\lambda_{a,t}^h h_{a,t}^{e*}\right)^{\eta^h}}{1 + \eta^h}}_{\text{disutility from hours worked}} + \underbrace{s_{a,t} \left(1 - \delta_{a+1,t+1}\right)}_{\text{probability of surviving with a job}} \underbrace{\beta \frac{Z_{a+1,t+1}^{lf}}{Z_{a,t}^{lf}}}_{\text{discount factor}} \underbrace{\left(\frac{1 - x_{t+1}^*}{x_{t+1}^*}\right) \left(\frac{\lambda_{a+1,t+1}^{lf}}{1 - q_{a+1,t+1}^{lf*}}\right)^{\eta^{lf}}}_{\text{saved disutility of search}} \right\}$$

In order to be consistent with the definition of structural employment used by the Ministry of Finance, we exogenize the augmented discount factor $\beta \frac{Z_{a+1,t+1}^{lf}}{Z_{a,t}^{lf}}$ in both the actual and structural participation condition when evaluating shocks to the model. This implies removing the dynamic effect of changes in real wages and marginal utility on the effective discount factor used in evaluating the effect of participation today on future employment.

8.1.4 Structural choice of hours

The hours decision in MAKRO is already structural, although users can insert an exogenous temporary gap between actual and structural hours worked:

$$h_{a,t}^e = h_{a,t}^{e*} + \epsilon_{a,t}^h = \frac{1}{\lambda_{a,t}^h} \left[\frac{1 - \tau_{a,t}^m}{Z_{a,t}^{\tau}}\right]^{\frac{1}{\eta^h}} + \epsilon_{a,t}^h$$

The labor utilization variable discussed in the production chapter provides a more general concept of effort, which may also capture unmeasured deviations in hours worked. Unlike hours worked, labor utilization varies endogenously in MAKRO and gives rise to gaps between structural and actual output. How we handle utilization in the structural model is covered in section 8.2.

8.1.5 Sector composition

MAKRO does not model a structural composition of demand across production sectors. Without an explicit structural demand side, the sectoral composition of output and employment is not well defined in the structural model. To align with the Ministry of Finance's estimation approach, public sector output and employment are assumed to be always structural. We do not decompose private sector structural output or employment across production industries; these quantities are only defined at the private-business aggregate. With the simplifying assumptions made for aggregating sectors for the purpose of wage bargaining, described in the labor market chapter, structural output and employment are independent of the decomposition of the the private sector.

8.1.6 Structural labor demand

In the previous sections, we have calculated a structural supply of labor taking the structural job finding rate, x^*, as given. In order to get an equilibrium x^* we also need a structural demand for labor and a matching function that connects workers and firms. Structural labor demand is specified at the private-business aggregate, in line with the sector-aggregation assumption used in wage bargaining.

First we relate the structural vacancy matching rate m^* to x^*. As in the model of actual employment, the number of job searchers finding a job must be equal to the number of vacancies filled:

$$x_t^* \left(n_t^{s,f*} + \sum_a n_{a,t}^{lf*} \right) = m_t^* v_t^*$$

To get a measure of structural vacancies, we start with the optimality condition for vacancy posting from the main model, evaluated at the private-business aggregate under the sector-aggregation assumption. The structural version replaces every endogenous quantity by its structural counterpart, uses the effective statutory corporate tax rate and the average payroll tax rate for private business, and drops wage rigidity and quadratic hiring costs (which vanish in any steady state). This yields a structural user-cost-to-wage ratio p_t^{L*}/w_t^* that satisfies

$$\frac{p_t^{L*}}{w_t^*} = \left(1 + \tau_t^L\right) \left[1 - \frac{\partial \left(\chi_t^* n_t^*\right)}{\partial n_t^*} + \hat{\beta}_{t+1} \frac{h_{t+1}^*}{h_t^*} \frac{\partial \left(\chi_{t+1}^* n_{t+1}^*\right)}{\partial n_t^*} \right]^{-1}$$

The hiring-cost derivatives are evaluated at within-period structural values of productivity, hours worked, separation, and matching rates, and the quadratic hiring-cost term is omitted because it vanishes in any steady state. We treat p_t^{L*}/w_t^* as a single variable in the structural block.

Using within-period structural values is consistent with the definition of structural employment as a quasi-steady state employment with a fixed labor market structure. In the main model an anticipated shock can affect firm hiring decisions even before the shock actually occurs, however, for structural employment there is no affect until the change is implemented after which the effect is fully incorporated immediately. Note however that very few shocks actually affect structural employment. On the supply side, the net benefit replacement rate determines households' economic incentives to seek work, and demographics and preferences directly affect supply. Changes in labor market structures such as hiring cost parameters, bargaining power, or efficiency of the matching function also affect structural employment. Changes in demand for labor on the other hand are largely irrelevant for structural employment however.

8.1.7 Structural wage

To close the model of structural employment we need a structural model of wage bargaining. We start from the wage equation derived in the labor market chapter under the sector-aggregation assumption set out there:

$$w_t = \left[\frac{1-\phi_t}{1-\phi_t\mu^o x_t}\right]\frac{p_t^L}{[1-\chi_t]\left[1+\tau_t^L\right]} + \{\text{wage rigidity}\}\,.$$

The structural counterpart differs in two ways. First, every endogenous quantity is replaced by its structural value (denoted by a star). Second, we drop the wage-rigidity term, which is zero in any steady state and would otherwise contribute nothing to a within-period structural relation. Cross-multiplying and dividing through by the structural wage w_t^* then yields a single equation in the structural user-cost-to-wage ratio:

$$1 = [1-\phi_t]\frac{\left(p_t^{L*}/w_t^*\right)}{[1-\chi_t^*]\left[1+\tau_t^L\right]} + \phi_t\mu^o x_t^*. \tag{8.1}$$

Structurally, the wage level is indeterminate; what matters for the structural labor market is the ratio between the marginal product of labor and the wage, which is pinned down by the structures of the labor market – primarily the bargaining power of the two sides, the vacancy-posting cost share χ_t^*, and the outside option of workers. We therefore treat the ratio p_t^{L*}/w_t^* as a single variable.

8.2 Structural output (gross value added)

We define structural output as the output that would be produced given structural employment, but conditional on the actual capital stock[71]. With structural employment given by the modified version of the labor market above, we can now model a measure of structural output. Specifically, we model structural aggregate gross value added, Y^*. To compute this

[71] See Vetlov, Hlédik, Jonsson, Kucsera & Pisani (2011) for a discussion on various definitions of potential output and the decision of whether or not to condition on the actual capital stock.

measure, the model employs an aggregate production function that takes actual capital and structural employment as inputs.

Aggregate structural gross value added combines structural output in private business with actual output in sectors assumed to have no output gap. At prices from the main model,

$$P_t^* Y_t^* = P_t^{\text{priv}} Y_t^{\text{priv}*} + \sum_{s \in \mathcal{S}^0} P_{s,t} Y_{s,t}$$

where $Y_t^{\text{priv}*}$ is structural gross value added in private business and $\mathcal{S}^0$ is the set of sectors with no output gap. This includes the public sector, extraction, shipping, agriculture, and housing. The public sector is exogenous in the main model and not directly affected by the business cycle; we additionally impose zero public sector employment and output gaps to match the Ministry of Finance. We do not decompose $Y_t^{\text{priv}*}$ or structural private employment across production branches.

Structural gross value added in private business is modeled directly as a Cobb-Douglas production function, in line with the assumptions that the Ministry of Finance uses when estimating the output gap:

$$Y_t^{\text{priv}*} = A_t \left(L_t^{\text{priv}*} \right)^{\alpha} \left(K_{t-1}^{\text{priv}} \right)^{1-\alpha}$$

where $L_t^{\text{priv}*}$ is structural effective labor input in private business and K_{t-1}^{priv} is the aggregate private-business capital stock from the main model, conditional on the actual capital stock as discussed above. The labor share parameter α is set to match the estimation method of the Ministry of Finance. Structural private employment in heads and hours is computed only at the private-business aggregate; public sector employment and hours are set equal to actual employment and hours, and the private-business aggregate is obtained residually from total structural employment.

The total factor productivity A is calculated endogenously such that the same Cobb-Douglas production function, but using actual rather than structural inputs, reproduces the output generated by the CES production structure of the main model (see the chapter on production). I.e. A is given by

$$Y_t^{\text{priv}} = A_t \left(u_t^{L,\text{priv}} L_t^{\text{priv}} \right)^{\alpha} \left(u_t^{K,\text{priv}} K_{t-1}^{\text{priv}} \right)^{1-\alpha} \tag{8.2}$$

Unlike for structural output, the inputs used in (8.2) depend on the variable rates of utilization. Factor utilization varies significantly over the business cycle and is the source of the endogenous productivity gap in MAKRO. Utilization is modeled such that is always 1 in the long run and we define structural output conditional on utilization being 1 structurally. Actual labor and capital inputs in (8.2) are the private-business aggregates from the main model, and utilization rates are aggregated at the same level.

9 Public production

Public sector output consists of all the different goods and services provided by the state, from education and health care, to the judicial system and defense, and to child care, elderly care, etc. This output, Y_G, is not exported. It is entirely consumed domestically. The vast majority of these services are paid for using tax revenues or using the intake from public debt issues.

A small share of public services are paid directly by private agents, as is the case for some co-payments for health care and education. These private payments are included in household consumption of services and we denote this private consumption of government supplied services as C_t^{Pub}. The government also makes direct investments, I_t^{Pub}, which is almost entirely publicly funded R&D and accounted for as a public production of investments in equipment[72]. The majority of public production is accounted for as government consumption of public goods and services, G_t^{Pub}, however.

Table 9.1 shows the input output matrix in MAKRO from a single year. The column in blue sums to the value of government consumption $P_t^G G_t$, while the row in red sums to the value of public production before taxes, $P_{\text{Pub},t}^Y Y_{\text{Pub},t}$. Their intersect, the cell in purple, is public production for government consumption and makes up the majority of both public production and government consumption.

9.1 Supply side

Public production, $Y_{\text{Pub},t}$, is produced just as private sector goods are, in the sense that it uses labor, capital equipment and structures, and intermediate inputs.

Most public services are not traded on a market, however, and therefore no market prices exist in the data. The accounting price of public production is instead the average cost of production rather than a market price. This is equivalent to saying that there are zero profits in the public sector.

The default setting in MAKRO is that all factors of public production (labor, capital equipment and structures, and intermediate inputs) are exogenous. The quantity of output produced is endogenous, but only reacts to shocks to the production factors.

The total cost of public production is the sum of the value of the inputs into production:

$$P_{\text{Pub},t}^Y Y_{\text{Pub},t} = w_t \rho_{\text{Pub},t} h_{\text{Pub},t} n_{\text{Pub},t} + \sum_k P_{k,\text{Pub},t}^I \delta_{k,\text{Pub},t-1} K_{k,\text{Pub},t-1} + P_{\text{Pub},t}^R R_{\text{Pub},t} + T_{\text{Pub},t}^{\text{NetY}} \tag{9.1}$$

where $w_t \rho_{\text{Pub},t} h_{\text{Pub},t} n_{\text{Pub},t}$ is the public sector payroll, and where following accounting standards for the public sector, the cost of public capital is entirely accounted for as

[72] Investments in research and development will likely be separated from investments in equipment in a future version of MAKRO.

Table 9.1
Input/output matrix

		R		C (private consumption)						G	I (investments)			X	Supply totals
		private	public	cBol	cBil	cEne	cVar	cTje	cTur	g	IB	IL	IM	xTot	
Domestic production	Agriculture (lan)	62,6	0,4			0,1	4,7	1,2		0,5		1,1	0,2	22,0	90,6
	Construction (byg)	67,0	8,6	3,7				0,7		7,4	227,5		0,7	33,6	349,3
	Energy provision (ene)	29,9	6,5	3,9		49,2	0,0	0,0				0,4	1,0	8,9	99,8
	Extraction (udv)	11,7	0,1	0,0			0,2	0,0				0,2	0,1	6,8	18,7
	Housing (bol)	0,0		226,3			0,0	0,0					0,1		226,4
	Manufacturing (fre)	221,2	9,5	1,8	0,1	0,0	54,2	1,7		1,1		3,2	54,7	446,2	793,8
	Public (off)							60,3		543,2			22,1		625,6
	Services, other (tje)	754,5	135,1	11,0	15,7	12,9	202,5	255,2		17,4	19,1	1,4	96,9	319,0	1840,6
	Shipping (soe)	18,8	1,0				0,0	2,2		0,2			0,2	220,4	242,9
Imports	Energy provision (ene)	41,5	0,6			5,6	0,0					0,2		6,0	53,9
	Extraction (udv)	10,6	0,0				0,0					0,1		0,0	10,5
	Manufacturing (fre)	240,1	27,4	1,3	32,3	2,1	106,1	4,6		5,3		4,4	68,6	179,8	672,1
	Services, other (tje)	371,1	18,4	0,0			3,6	15,2	23,6	0,2	6,4		20,8	6,1	465,4
Demand totals		1829,1	207,5	247,9	48,1	69,9	371,4	341,2	23,6	575,4	252,9	8,3	265,5	1248,8	5489,5

2021 input/output table in current prices before taxes (bn. DKK). Columns representing demand for exports and intermediate inputs have been aggregated. The red row is government production. The blue column is government consumption.

depreciation[73]. Net production taxes and subsidies, $T^{\text{NetY}}_{\text{Pub},t}$, are added explicitly rather than being included in the payroll and cost of capital.

Given the total cost of public production, the issue is how to measure separately the quantity of public output, $Y_{\text{Pub},t}$, and its price $P^{Y}_{\text{Pub},t}$. In our model of the private sector we solve this problem using a theory of production. This is materialized in a (CES) production function that describes how the quantities of inputs are organized to generate units of output. The output price is then a by-product of this theory and of profit maximization. For the public sector, we generally do not apply a standard theory of production, but instead apply the *input method*.

9.1.1 Input method

We have separate measures of the price and quantity of each factor input in public production. We can measure investment, capital stocks, employment, and quantities of intermediate inputs used. From these, we create a Paasche price index of the underlying factor prices (set to 1 in the base year) and a corresponding Laspeyres measure of the quantity, so that in addition to eq. (9.1) we have

$$\begin{gathered} P^{Y}_{\text{Pub},t-1} Y_{\text{Pub},t} = \gamma_t w_{t-1} \rho_{\text{Pub},t-1} h_{\text{Pub},t} n_{\text{Pub},t} \\ + \sum_k P^{I}_{k,\text{Pub},t-1} \delta_{k,\text{Pub},t-1} K_{k,\text{Pub},t-1} + P^{R}_{\text{Pub},t-1} R_{\text{Pub},t} + \frac{T^{\text{NetY}}_{\text{Pub},t-1}}{Y_{\text{Pub},t-1}} Y_{\text{Pub},t} \end{gathered} \tag{9.2}$$

In the data, public employees are differentiated by educational attainment etc., and changes in the average public wage may therefore reflect changes in the composition of public employees rather than actual changes in wages for the individual public employee. E.g. the average public wage increases if the share of highly educated workers increases, even if the wage pr. highly educated worker remains constant. An additional exogenous term, γ_t, is included with the lagged wage to capture this effect in the baseline forecast. In addition, in years covered by data, this term also captures differences between the input method and the national accounting estimates of public production, which attempts to measure actual output (number of surgeries etc.).[74]

9.1.2 Public employment

The expenditure on labor by the public sector consists of wages paid pr. hour, $w_t \rho_{\text{Pub},t}$, times hours pr. employee, $h_{\text{Pub},t}$, times the number of employee, $n_{\text{Pub},t}$. Payroll taxes, τ^{L}_{t}, are disregarded here as they are transfers from the state to itself. The wage expenditure also disregards vacancy posting costs as these are a component of the user cost of labor, which is

[73] In years covered by data, we include residual terms in the depreciation rates and prices used to compute the value of public capital depreciation in order to match the national accounts data. Public capital depreciation would otherwise differ slightly from the national accounts data due to composition effects in the prices of capital and investment which affect the chain indices used to calculate prices.

[74] The chain index method of accounting for public production has the unfortunate feature that a shock which leaves all factors of public production completely unchanged can still affect the Laspeyres measure of output through price changes. For a transitory shock, the effect is negligible, but for a permanent shock the effect can be difficult to ignore, as the effect accumulates over time. The problem also exists for other chain indices, but is exacerbated here by the fact that capital and intermediate inputs generally follow productivity growth, while hours worked do not. For shocks to the model, which do not affect any public sector factor of production, users are therefore advised to also exogenize government consumption, G_t, rather than using eq. (9.2).

not considered in the input method of accounting for the public sector. I.e. since we merely measure costs, rather than apply a standard theory of production, it does not matter whether the labor is spent as hiring costs or used in production. The hourly wage, $w_t \rho_{\text{Pub},t}$, is detailed in the labor market chapter.

Public employment in hours is exogenous by default in MAKRO. In the baseline forecast, the ratio between public capital and output is fixed, as well as the ratio between the public payroll and expenditure on intermediate purchases (we go into detail on these ratios below). With the public K/Y and L/R ratios fixed in the baseline, public employment is determined residually given a baseline demand for public production. The baseline demand for public production is covered in section 9.2.

9.1.3 Public investments

Public investments as real quantities are exogenous by default in MAKRO, and it is up to the user to shock public investments explicitly for a balanced shock to public production. Public sector capital (equipment and structures) obeys the standard law of motion

$$K_{k,\text{Pub},t} = (1 - \delta_{k,\text{Pub},t}) K_{k,\text{Pub},t-1} + I_{k,\text{Pub},t}$$

In the baseline forecast, public investments are determined residually as we forecast the public sector capital stock directly. Public sector capital in current investment prices is forecasted as a constant share, $\hat{\alpha}^K_{k,t}$, of a weighed average of public and private sector gross value added. That is, we add the restriction

$$P^I_{k,t} K_{k,t} = \hat{\alpha}^K_{k,t} \left(0.7 \cdot \left(P^Y_{\text{Pub},t} Y_{\text{Pub},t} - P^R_{\text{Pub},t} R_{\text{Pub},t}\right) + 0.3 \cdot X_t\right)$$

where X_t is a moving average of private sector value added. $\hat{\alpha}^K_{k,t}$ is constant in the baseline forecast, but is endogenous by default for shocks to the model as public investments are fixed.

New, direct, and indirect investments In addition to distinguishing between capital structures and equipment, public investments are also classified in three other categories: *new*, *direct*, and *indirect* investments. In this section we document how these categories are allocated between equipment and structures and how we forecast these categories.

The majority of public investments are categorized as *new investments*, and we divide these between both capital types with a share parameter, μ^{new}_t. Direct investments, which consist almost entirely of publicly funded R&D are allocated to investments in equipment. Finally, indirect investments are purchases of existing capital and are entirely allocated to structures. In current prices, we have the following accounting identities:

$$\text{public investments}_t = \text{new investments}_t + \text{direct investments}_t + \text{indirect investments}_t$$

$$\text{public investment in structures}_t = \mu_t^{new} \cdot \text{new investments}_t + \text{indirect investments}_t$$

$$\text{public investment in equipment}_t = (1 - \mu_t^{new}) \cdot \text{new investments}_t + \text{direct investments}_t$$

Indirect investments are modeled as a constant share of aggregate gross value added

$$\text{indirect investments}_t = \mu_t^{indirect} \cdot \left(P_t^Y Y_t - P_t^R R_t\right)$$

Note that the separation between indirect and new investments is purely accounting and has no effect on any behavior in MAKRO.

Direct investments are modeled as a fixed share of public production in real quantities.

$$\text{direct investments}_t = P_{\text{Pub},t}^Y \cdot \mu_{iM,t}^{\text{Pub}} \cdot Y_{\text{Pub},t}$$

New investments are determined residually, given total public investments.

9.1.4 Intermediate input purchases

The aggregate quantity of intermediate inputs used in public production is exogenous by default in MAKRO.

Intermediate inputs in the public sector are sourced from many production sectors, both domestic and foreign, in exactly the same way as other demand components and is covered in the input/output chapter.

In the baseline forecast we assume a constant ratio, μ_t^{LR}, between the public payroll and expenditures on intermediate inputs:

$$\mu_t^{LR} = \frac{w_t \rho_{\text{Pub},t} h_{\text{Pub},t} n_{\text{Pub},t}}{P_{\text{Pub},t}^R R_{\text{Pub},t}}$$

For a generic shock to government consumption, the user can exogenize this ratio in order to determine a proportional change in public sector intermediate purchases.

9.2 Demand for public production

Public production, $Y_{\text{Pub},t}$, is given in the input/output system as the sum of three demand components: government consumption (of public services) G^{Pub}, private consumption of public services C^{Pub}, and public direct investments, I^{Pub}.

$$Y_{\text{Pub},t} = C_t^{\text{Pub}} + G_t^{\text{Pub}} + I_t^{\text{Pub}}$$

As we assume that the same price (before taxes) is paid independent of how output is consumed, this identity holds both for real quantities and in current prices.

9.2.1 Private consumption of public services and public direct investments

We model the private consumption of public services as proportional to government consumption:

$$C_{c,t}^{\text{Pub}} = \mu_{c,t}^{\text{Pub}} G_t$$

The underlying assumption is that the structure of public services are fixed, such that co-payments remain a constant share except if the user shocks these structures explicitly.

Similarly, public direct investments (R&D) are a fixed share of public production:

$$I_{k,t}^{\text{Pub}} = \mu_{k,t}^{\text{Pub}} Y_{\text{Pub},t}$$

These two assumptions imply that public production is fixed except for shocks that affect the government consumption, as opposed to moving endogenously with changes in private sector demand through constant input/output shares.

9.2.2 Government consumption of public production

Total government consumption G_t is the sum of government consumption of public output plus government consumption of private output, including imports (the blue column in table 9.1):

$$P_t^G G_t = P_{G,Pub,t}^{Y\tau} G_t^{Pub} + \text{Private inputs} = \sum_s \left(P_{G,s,t}^{Y\tau} Y_{G,s,t} + P_{G,s,t}^{M\tau} M_{,G,s,t} \right)$$

The sourcing of government consumption from different sectors, domestic and foreign, works in exactly the same way as other demand components and is covered in the input/output chapter. Given a desired total government consumption, the input/output system delivers a demand for public production, or vice versa given a total public production, we can calculate total government consumption as a residual.

As all factors of production are fixed by default, the supply of public production $Y_{\text{Pub},t}$ is also effectively fixed, and the demand for government consumption is determined residually.

9.2.3 Baseline government consumption

We decompose the costs of government consumption in two parts: the depreciation costs of capital and the remainder:

$$P_t^G G_t = \sum_k \delta_{\text{Pub},k,t} P^I_{\text{Pub},k,t} K_{\text{Pub},k,t} + \mu_t^G N_t^G \rho_t w_t$$

In the baseline, the remainder (government consumption net of depreciations), follow a fixed *government service level* parameter, μ_t^G, times an exogenous demographics based index of demand of government services, N_t^G, and finally the average wage $\rho_t \cdot w_t$ pr. hour, in line with the forecast methodology of the Ministry of Finance. The demographics index is an important exogenous input to the model baseline and based on more detailed demographic projections (befolkningsregnskab).

Table 9.2
Public production variables

Documentation	Code	Definition
$Y_{\text{Pub},t}$	qY['off',t]	Public production
$h_{\text{Pub},t}n_{\text{Pub},t}$	hL['off',t]	Public sector employment in hours worked
$K_{k,\text{Pub},t}$	qK[k,'off',t]	Public sector capital of type k, ultimo
$R_{\text{Pub},t}$	qR['off',t]	Intermediate inputs to public production
G_t	qG['g',t] = qG[gTot,t]	Government consumption
G_t^{Pub}	qIO['g','off',t]	Inputs from public production to government consumption
C_t^{Pub}	qIO['cTje','off',t]	Inputs from public production to private consumption of services
I_t^{Pub}	qIO['iM','off',t]	Inputs from public production to capital equipment
$\delta_{k,\text{Pub},t}$	rAfskr[k,'off',t]	Depreciation rate of public capital of type k
$\rho_{\text{Pub},t}$	qProd['off',t]	Labor units pr. hour in public sector
$I_{k,\text{Pub},t}$	qI_s[k,'off',t]	Public sector investments of type k
$Y_{G,s,t}$	qIOy['g',s,t]	Domestic inputs from sector s to government consumption
$M_{G,s,t}$	qIOm['g',s,t]	Imports from sector s to government consumption
$P^Y_{\text{Pub},t}$	pY['off',t]	Deflator for public production
w_t	pW[t]	Wage per unit of productive labor
$P^I_{k,\text{Pub},t}$	pI_s[k,'off',t]	Deflator for public sector investments of type k
$P^R_{\text{pub},t}$	pR[''off',t]	Deflator for intermediate inputs to public production
P^G_t	pG[gTot,t]	Deflator for government consumption.
$P^{Y\tau}_{G,s,t}$	pIOy['g',s,t]	Price of domestic deliveries from s to 'g' including taxes
$P^{M\tau}_{G,s,t}$	pIOm['g',s,t]	Price of imported deliveries from s to 'g' including taxes
$T^{\text{NetY}}_{\text{Pub},t}$	vtNetY['off',t]	Value of net production taxes in public production

9.3 Appendices - Public Production

9.3.1 Productivity Growth

It is assumed the there is no labor augmenting technological progress in the public sector. A simple way to understand the consequences of this fact is to work as if public production happened through a Cobb-Douglas production function with inputs (K_b, K_m, L, R):

$$Y_t = K_{b,t}^{\alpha_b^K} K_{m,t}^{\alpha_m^K} (A_t L_t)^{\alpha^L} R_t^{\alpha^R}$$

In such a case the price would be the variable recovered through the zero profit condition, and this price would be

$$P_t = A_t^{-\alpha^L} \left(\frac{P_{b,t}^K}{\alpha_b^K}\right)^{\alpha_b^K} \left(\frac{P_{m,t}^K}{\alpha_m^K}\right)^{\alpha_m^K} \left(\frac{P_t^L}{\alpha^L}\right)^{\alpha^L} \left(\frac{P_t^R}{\alpha^R}\right)^{\alpha^R}$$

On a balanced growth path all input prices or user costs grow with the inflation rate, except for the user cost of labor, which increases with the inflation rate plus the Harrod neutral growth rate g_ξ. This implies for a Cobb-Douglas output price:

$$\frac{P_t}{P_{t-1}} = (1+\pi)^{\alpha_b^K + \alpha_m^K + \alpha^R} \left(\frac{(1+\pi)(1+g_\xi)}{(1+g_\xi)}\right)^{\alpha^L} = 1+\pi$$

It is, however, assumed that there is no productivity growth in the public sector. This implies that its price will grow with a higher rate, namely

$$\frac{P_t^0}{P_{t-1}^0} = (1+\pi)^{\alpha_b^K + \alpha_m^K + \alpha^R} \left(\frac{(1+\pi)(1+g_\xi)}{1}\right)^{\alpha^l} = (1+\pi)(1+g_\xi)^{\alpha^L}$$

where the contribution of the growth rate of technology on the labor price is weighed by the labor share.

This is captured in the price index of public production by adding the growth of technology in the denominator as follows

$$P_t^0 = P_{t-1}^0 \frac{\sum_{i\in(b,m)} P_{i,t}^I K_{i,t} + P_t^R R_t + \left(P_t^L / \cancel{A_t}\right)\left(\cancel{A_t} L_t\right)}{\sum_{i\in(b,m)} P_{i,t-1}^I K_{i,t} + P_{t-1}^R R_t + \left(P_{t-1}^L / A_{t-1}\right)\left(A_t L_t\right)}$$

and this works in the desired way because the technology factor grows at a constant rate in all sectors except for the public sector where it is constant.

10 Government

This chapter provides an overview of government revenues and expenditures. Several items on the expenditure side of the balance sheet are exogenous or obey exogenous relationships, such as fixed ratios to population or GDP. The same is true on the revenue side, as much of this side of the balance sheet amounts to determining the realized average tax rate on a particular item.

A couple of basic relationships are helpful to put forward here. First, the government budget surplus is the primary budget surplus plus net interest income,

$$\text{Surplus} = \text{Primary Surplus} + \text{Net Interests}$$

and the primary surplus is the difference between revenues and expenditures (other than interest payments).

$$\text{Primary Surplus} = \text{Revenues - Expenditures}$$

Revenues and expenditures are described in the next two sections. Section 10.3 accounts for net interests, and finally, after detailing the balance sheet, we define the fiscal sustainability indicator in section 10.4.

10.1 Revenue

Government revenue is the sum of direct taxation, indirect taxation and other government revenues:

$$\text{Revenues} \equiv T_t = T_t^{Direct} + T_t^{Indirect} + T_t^{Other}$$

Direct taxes comprise around 60% of total tax revenues and are described in subsection 10.1.1, which covers many aspects of the Danish income tax system. Indirect taxes consist mainly of duties, VAT, and production taxes, as described in subsection 10.1.3. Indirect taxes make up around 30% of tax revenues. The remaining taxes are described in subsection 10.1.4.

Regarding notation, the letter y stands for income and appears in different objects, tax rates are denoted by τ, and tax revenues are given by the capital letter T. For example, the sector-specific corporate tax rate is called $\tau_{s,t}^{Corp}$ and brings in the total revenue $T_{s,t}^{Corp}$. Tax rates in the text, τ, correspond to tax rates in the code $t \times f$, where f is an adjustment variable to fit the data. These adjustments help match observed average tax rates, given the rate determined in the tax law. The adjustment factor is sometimes unnecessary and set to 1. Tax rates and adjustment factors are typically exogenous in MAKRO - if not it is explicitly mentioned.

Where applicable, variables such as T_t^{Income} represent sums over all cohorts, while corresponding variables with an age subscript, $T^{Income}_{a,t}$, represent cohort averages. The two variables are related by $T^{Income}_t = \sum_a N_{a,t} T^{Income}_{a,t}$.[75]

10.1.1 Direct taxes

Direct taxation is modeled closely after the Danish income tax law.[76] Economic and demographic movements affect the tax burden such that the relationship between direct taxation and the total income level is not constant, and therefore we need a flexible modeling of the tax system.

Direct taxes consist of income taxes T^{Income}_t, labor market contributions (AM bidrag)[77], T^{AM}_t, corporate taxation, T^{Corp}_t, taxation on the return on investments in pension funds, T^{PAL}_t, weight duties on cars, T^{Weight}_t, and other personal income taxation, $T^{PersRest}_t$:

$$\begin{aligned} T^{Direct}_t = \; & T^{Income}_t + T^{AM}_t + T^{Corp}_t + T^{PAL}_t \\ & + T^{Weight}_t + T^{PersRest}_t \end{aligned}$$

Income tax revenues are given by

$$\begin{aligned} T^{Income}_t = \; & T^{Muncipal}_t + T^{Bot}_t + T^{Equity}_t + T^{Top}_t \\ & + T^{Property}_t + T^{Business}_t + T^{Deceased}_t \end{aligned}$$

where the components are local (municipal) income taxes, income taxes from the bottom income bracket, taxes on capital income from equity, income from the top income bracket, property taxes, taxes on small businesses that do not pay corporate tax, and taxes on the deceased (as households can still have income subject to taxation in the year they die). All these revenues are calculated for each cohort of households.

The municipal tax, bottom bracket tax, and top bracket tax revenues are given by:

$$T^{Muncipal}_{a,t} = \tau^{Muncipal}_{a,t} \left(y^{Taxable}_{a,t} - AL^{Personal}_{a,t} \right)$$

$$T^{Bot}_{a,t} = \tau^{Bot}_t \left(y^{Personal}_{a,t} + y^{NetCap^+}_{a,t} - AL^{Personal}_{a,t} \right)$$

$$T^{Top}_{a,t} = \tau^{Top}_t \left(y^{Personal}_{a,t} + y^{NetCap^+}_{a,t} \right) \alpha^{Top}_{a,t}$$

The municipal tax is based on taxable income, $y^{Taxable}_{a,t}$, while the bottom and top bracket taxes are based on personal income, $y^{Personal}_{a,t}$, plus positive[78] net income from bonds and deposits,

---

[75] The only exceptions for tax revenues in this chapter are the taxes on the deceased and property tax. However, it does not apply in general as, in some cases, the base is the start-of-year population, and the flow of the deceased needs to be accounted for.

[76] See http://www.skm.dk/skattetal/beregning/skatteberegning/skatteberegning-hovedtraekkene-i-personbeskatningen-2017

[77] AM Bidrag is a tax of 8%, which all employees and the self-employed must pay each month on their wages. Employers ensure that the labor market contribution is automatically deducted from salary after ATP and any own pension contribution have been deducted, after which the other taxes are deducted.

[78] Positive means that the net income form bonds and deposits is conditional on being positive and above a

$y_{a,t}^{NetCap^+}$. Only income above a certain threshold is taxed at the 3 relevant tax rates. For the municipal and the bottom bracket taxes, a personal allowance, $AL_{a,t}^{Personal}$, is subtracted from the base. For top bracket taxes, $\alpha_{a,t}^{Top}$ controls the share of personal income effectively taxed at the top rate. The income terms and allowances are described in detail in subsection 10.1.2.

The taxes on income generated by financial stocks are given by:

$$\underbrace{T_{a,t}^{Equity} N_{a,t}}_{\text{Revenue from survivors}} = \tau_t^{Equity} \left(\underbrace{\sum_{i \in \text{Equity}} r_{i,t}^{Div} A_{i,a-1,t-1}}_{\text{Dividends (pr. household)}} + \underbrace{\alpha_t^{Gains} C_t^{AccGains}}_{\text{Aggregate capital gains}} \underbrace{\frac{\sum_{i \in \text{Equity}} A_{i,a-1,t-1}}{\sum_{i \in \text{Equity}} A_{i,t-1}}}_{\text{Household share of total}} \right) \underbrace{s_{a-1,t-1} N_{a-1,t-1}}_{\text{Surviving population}}$$

where $A_{i,a,t}$ is the households financial assets of type i, here either domestic or foreign equities. Both dividends and realized capital gains are subject to taxes on equity income. Only the surviving population pays regular equity taxes, as the deceased are instead taxed through a special tax on the deceased.

Unrealized capital gains are a stock, $C_t^{AccGains}$, of accumulated gains and *losses* from price changes on assets which have not yet been realized and taxed. We model the realization of these gains as a fixed share, α_t^{Gains}, of the unrealized stock. The law of motion of the unrealized-capital-gains stock is

$$C_t^{AccGains} = C_{t-1}^{AccGains} - \alpha_t^{Gains} C_t^{AccGains} + \sum_{i \in \text{Equity}} r_{i,t}^{Reval} A_{i,t-1}$$

where $r_{i,t}^{Reval}$ is the price change of asset i.[79] This formulation implies that capital gains are gradually taxed with an average realization time depending on α_t^{Gains}.

Property taxes follow the value of the start-of-period stock of owner-occupied housing, $D_{a-1,t-1}$:[80]

$$T_{a,t}^{Property} N_{a,t} = \tau_t^{Property} D_{a-1,t-1} s_{a-1,t-1} N_{a-1,t-1}$$

The business tax follows earnings before taxes, EBT_t. It is distributed among cohorts according to their wage income assuming that business income is distributed as wage income:

$$T_{a,t}^{Business} = \tau_t^{Business} EBT_t \frac{W_{a,t}}{W_t}$$

where $W_{a,t}$ denotes the wages per person paid to the households of age a and W_t is the total

certain level.

[79] A revaluation is a capital gain that is not realized (where assets change prices but are not traded). A capital gain occurs when the asset is traded.

[80] Note that in the current version of MAKRO $T_t^{Property} = T_{a,t}^{Property} N_{a,t} + \tau_t^{Property} D_{a-1,t-1} (1 - s_{a-1,t-1}) N_{a-1,t-1} = \sum_a \tau_t^{Property} D_{a-1,t-1} N_{a-1,t-1}$. That is property taxes on the deceased (not a part of taxes on the deceased) must also be paid. These are paid through lower inheritance. This will be changed in a future version of MAKRO.

of the cohort.[81]

Taxation of the deceased primarily consists of taxes on capital income of the deceased. It follows the base for equity taxes and other capital income for those that do not survive:

$$T_{a,t}^{Death} = \tau_t^{Death} \left(\underbrace{\sum_{i \in \text{Equity}} r_{i,t}^{Div} A_{i,a-1,t-1}}_{\text{Dividends}} + \underbrace{\alpha_t^{Gains} C_t^{AccGains} \frac{\sum_{i \in \text{Equity}} A_{i,a-1,t-1}}{\sum_{i \in \text{Equity}} A_{i,t-1}}}_{\text{Capital gains}} \right. \\ \left. + \underbrace{r_{a,t}^{NetCap^+} \sum_{i \in \text{Bonds \& Deposits}} r_{i,t}^{Interest} A_{i,a-1,t-1}}_{\text{Interests}} \right)$$

The first two terms of the base refer to equity taxes, and the last refers to positive net capital income, which is described in the subsection "Income terms and allowances". The aggregate revenue from taxes on the deceased is given by:

$$T_t^{Death} = \sum_a T_{a,t}^{Death} \left(1 - s_{a-1,t-1}\right) N_{a-1,t-1}$$

Labor market contributions (AM Bidrag) are modeled as:

$$T_{a,t}^{AM} = \tau_t^{AM} W_{a,t} \left(\frac{W_t - T_t^{CivilServants}}{W_t} \right)$$

They depend on the tax rate and on wages per person (not per employee) adjusted for pension contributions to civil servants' pensions.[82]

Corporate taxes are given by:

$$T_t^{Corp} = \sum_{s \in Private} T_{s,t}^{Corpx} + T_t^{NorthSea}$$

which is the sum of corporate tax revenue from the private sectors, $T_{s,t}^{Corpx}$, plus a tax for oil and gas extraction in the North Sea, $T_t^{NorthSea}$. The corporate tax excluding extraction is levied on earnings before taxes, EBT_t :

$$T_{s,t}^{Corpx} = \tau_t^{Corp} EBT_{s,t}$$

while tax revenues from oil and gas extraction is based on earnings before taxes, interests, and depreciations in the extraction sector and given by:

$$T_t^{NorthSea} = \tau_t^{CorpNorth} EBITDA_{s \in \text{Extraction},t}$$

[81] In the notation of the labor market chapter it is defined as $W_{a,t} \equiv \frac{\rho_{a,t}^e h_{a,t}^e n_{a,t}^e w_t}{N_{a,t}}$ and $W_{a,t} \equiv \sum_a W_{a,t} N_{a,t} = \sum_a \rho_{a,t}^e h_{a,t}^e n_{a,t}^e w_t$.

[82] The last term is modifying the age dependent wage to be net of civil servants' contribution. This is modeled with the extra term as this contribution is not age dependent.

Pension funds pay tax on the return on their financial assets (interest on bonds, dividends and capital gains on stocks):

$$T_t^{PAL} = \tau_t^{PAL} \sum_{i \in \text{Pensions}} \left(r_{i,t}^{Interest} + r_{i,t}^{Reval} \right) A_{i,t-1}$$

where the rate of return is the sum of interests (including dividends), $r_{i,t}^{Interest}$, and revaluations, $r_{i,t}^{Reval}$, of the pension portfolio.

The weight charge on cars, $T_{a,t}^{Weight}$, is calculated per person by an implicit tax rate times the stock of privately owned cars, $Cars_t$, distributed by age according to non-housing consumption, $C_{a,t}$:[83]

$$T_{a,t}^{Weight} = \tau_t^{Weight} Cars_{t-1} \frac{C_{a,t}}{C_t}$$

Finally, other (residual) personal income taxes are given by the tax on income received from capital pensions, and a further term divided according to personal income times an implicit tax rate:

$$T_{a,t}^{PersRest} = \tau_t^{CapPension} y_{p \in \text{Capital pension},a,t}^{PY} + y_{a,t}^{Personal} \tau_t^{PRNCP}$$

where tax on income received from capital pensions is given by a fixed rate times the pension income from capital pensions, $y_{p \in \text{Capital Pension},a,t}^{PY}$.

10.1.2 Income terms and allowances

Personal income is given by:

$$y_{a,t}^{Personal} = \frac{W_{a,t}}{N_{a,t}} - T_{a,t}^{AM} + y_{a,t}^{GTax} + y_{p \in \text{Taxable excl. capital pension},a,t}^{PY} - y_{p \in \text{Taxable incl. capital pension},a,t}^{CP} + y_{a,t}^{PersonalRest}$$

which is wage income per person, $\frac{W_{a,t}}{N_{a,t}}$, excluding labor market contributions, $T_{a,t}^{AM}$, plus taxable income transfers, $y_{a,t}^{GTax}$, defined below in the section on government expenses, plus (received) pension income subject to income taxation, $y_{p \in \text{Taxable excl. capital pension},a,t}^{PY}$, minus pension payments to the two tax-deductible types of pensions, $y_{p \in \text{Taxable incl. capital pension},a,t}^{CP}$.[84] Pensions are discussed in the household chapter. Finally a residual item, $y_{a,t}^{PersonalRest}$, ensures that personal income by age matches administrative data.

Taxable income adds net capital income, $y_{a,t}^{NetCapital}$, and subtracts a number of allowances (AL) defined below:

[83] To make the model more consistent we could have age specific car stocks. Then the distribution of weight tax on age could be consistent to the prior car consumption by age. As we do not have car consumption divided by age we assume it to be proportional to overall consumption, thus sparing the extra book keeping by age, and distribute the tax according to non-housing consumption.

[84] Income received from capital pensions is not taxed as personal income, but with an independent tax rate. Payments into capital pensions are tax deductible.

$$y_{a,t}^{Taxable} = y_{a,t}^{Personal} + y_{a,t}^{NetCapital} - AL_{a,t}^{EITC} - AL_{a,t}^{Unemp} - AL_{a,t}^{EarlyRet} - AL_{a,t}^{Other}$$

Net Capital Income of an average person of age a is the difference

$$y_{a,t}^{NetCapital} = y_{a,t}^{Cap^+} - y_{a,t}^{Cap^-}$$

where positive capital income is the return on household nominal deposits and bonds $\sum_{i \in \text{Bonds \& Deposits}} r_{i,t}^{interest} A_{i,a,t}$ and negative capital income consists of interest payments on nominal bank debt and mortgage debt $\sum_{i \in \text{Debt \& Mortgages}} r_{i,t}^{interest} L_{i,a,t}$.

Positive and negative capital income is given by:

$$\underbrace{y_{a,t}^{Cap^+} N_{a,t}}_{\text{Income to survivors}} = \underbrace{\sum_{i \in \text{Bonds \& Deposits}} r_{i,t}^{Interest} A_{i,a-1,t-1}}_{Interests} \underbrace{s_{a-1,t-1} N_{a-1,t-1}}_{\text{Surviving population}}$$

$$\underbrace{y_{a,t}^{Cap^-} N_{a,t}}_{\text{Payments by survivors}} = \underbrace{\sum_{j \in \text{Debt \& Mortgages}} r_{j,t}^{Interest} L_{j,a-1,t-1}}_{Interests} \underbrace{s_{a-1,t-1} N_{a-1,t-1}}_{\text{Surviving population}}$$

where $A_{i,a,t}$ and $L_{j,a,t}$ are the households' financial assets and liabilities of type i and j. The deceased do not pay regular income taxes on capital income. These are instead part of *taxes on the deceased*.

All capital income is part of taxable income and enters the tax base for municipal taxation. However, only positive net capital income (above a certain threshold, $y_{a,t}^{NetCapital} > \underline{y} > 0$) is part of the tax base for bottom and top taxation[85]:

$$y_{a,t}^{NetCap^+} = \left(y_{a,t}^{NetCapital} > \underline{y} > 0\right) \equiv r_{a,t}^{NetCap^+} y_{a,t}^{Cap^+}$$

The potential personal allowance is the same for every (adult) person and follows the wage indexation of transfers (satsregulering, s_t^{reg}, described in subsection 10.1.3). The actual average personal allowance used is, however, not the same for all cohorts as some (few) persons do not have any income[86]:

$$AL_{a,t}^{Personal} = s_t^{reg} \bar{AL}_{a,t}^{Personal}$$

where $\bar{AL}_{a,t}^{Personal}$ is set to match data.

The earned income tax credit (Beskæftigelsesfradrag, EITC) is an allowance for people in employment. It is a percentage of income up until a limit. It has the properties of a negative marginal tax for people with low income and a negative lump sum tax for people with high income. It is treated as a negative marginal tax, but with a tax rate equaling the average

[85] We are looking at micro data for an accurate measure and until then we set $r_{a,t}^{NetCap^+} = 0.5$.
[86] The personal allowance can be used by a spouse if a person has no income (and is married). This effect is not captured in the model.

relative allowance. It could be distributed on age groups according to administrative data, but in this model version it is assumed to be the same for all age groups. This means the total tax credit can be calculated as the average allowance rate times wages:

$$AL_{a,t}^{EITC} = \tau_t^{EITC} W_{a,t}$$

The allowance for contribution to unemployment insurance, $AL_{a,t}^{Unemp}$, is proportional to the contributions paid towards unemployment insurance.[87]:

$$AL_{a,t}^{Unemp} = \frac{Cont_{a,t}}{Cont_t} A2C_t^{Unemp} Cont_t^{Unemp}$$

and similarly, the allowance for contribution to early retirement (Efterløn), $AL_{a,t}^{EarlyRet}$, is proportional to contributions toward early retirement:

$$AL_{a,t}^{EarlyRet} = \frac{Cont_{a,t}}{Cont_t} A2C_t^{EarlyRet} Cont_t^{EarlyRet}$$

We have data for AL_t^{Unemp} and $Cont_t^{Unemp}$. The age decomposition follows the age distribution from total contributions (as only total contributions, $Cont_{a,t}$, have an age distribution), which is simply proportional to wage income by age.

Other allowances include allowances for transport, clothes etc. These are primarily related to employment and therefore modeled as a function of the employment rate by age:

$$AL_{a,t}^{Other} = \bar{AL}_t^{Other} s_t^{reg} \frac{n_{a,t}^e}{N_{a,t}}$$

where $\bar{AL}_t^{Other}$ is exogenous and set to match data.

10.1.3 Indirect taxation

The tax revenue from indirect taxes consist of the revenues from value added taxes, T_t^{VAT}, excise duties, T_t^{Duty}, duties from car sales (registreringsafgift), T_t^{Reg}, production taxes, $T_t^{Production}$, and the difference between customs taxes (taxes on imported goods), T_t^{Cus}, and indirect taxes to the EU, T_t^{EU}.

$$T_t^{Indirect} = T_t^{VAT} + T_t^{Duty} + T_t^{Reg} + T_t^{Production} + T_t^{Cus} - T_t^{EU}$$

Value added taxes, excise duties, and customs are all taxes on demand inputs from the production sectors. Customs are added first, but only to imports:

$$T_t^{Cus} = \sum_d \sum_s \tau_{d,s,t}^{Cus} P_{s,t}^M M_{d,s,t}$$

[87] Allowances include contributions and administration cost. Therefore the ratios of allowance to contribution $A2C_t^{Unemp}$ and $A2C_t^{EarlyRet}$ can be above 1.

where $\tau^{Cus}_{d,s,t}$ is the exogenous custom rate, $M_{d,s,t}$ are imports from sector s to demand group d, and $P^{M}_{s,t}$ is the corresponding price. Excise duties are added on top of customs:

$$T^{Duty}_{t} = \sum_{d}\sum_{s}\left[\tau^{DutyY}_{d,s,t} P^{Y}_{s,t} Y_{d,s,t} + \tau^{DutyM}_{d,s,t}\left(1+\tau^{Cus}_{d,s,t}\right) P^{M}_{s,t} M_{d,s,t}\right]$$

where $\tau^{DutyY}_{d,s,t}$ and $\tau^{DutyM}_{d,s,t}$ are the exogenous duty rates, $Y_{d,s,t}$ is domestic input from sector s to demand group d, and $P^{Y}_{s,t}$ is the corresponding price. Value added taxes are added on top of duties (except those from car sales) and subsidies:

$$T^{VAT}_{t} = \sum_{d}\sum_{s}\left[\begin{array}{c} \tau^{VATY}_{d,s,t}\left(1+\tau^{DutyY}_{d,s,t} - s^{Y}_{d,s,t}\right) P^{Y}_{s,t} Y_{d,s,t} \\ +\tau^{VATM}_{d,s,t}\left(1+\tau^{DutyM}_{d,s,t} - s^{M}_{d,s,t}\right)\left(1+\tau^{Cus}_{d,s,t}\right) P^{M}_{s,t} M_{d,s,t} \end{array}\right]$$

where $s^{Y}_{d,s,t}$ and $s^{M}_{d,s,t}$ are exogenous subsidy rates. Note that duty and VAT rates vary for imported and domestic inputs. The tax structure does not allow for such discrimination. It is a product of aggregation as imported and domestic inputs have different composition and hence different aggregate tax rates even on input/output cell level.

Duties from car sales are added after VAT and are modeled as:

$$T^{Reg}_{t} = \sum_{d}\sum_{s}\left[\begin{array}{c} \tau^{Reg}_{d,s,t}\left(1+\tau^{VATY}_{d,s,t}\right)\left(1+\tau^{DutyY}_{d,s,t} - s^{Y}_{d,s,t}\right) P^{Y}_{s,t} Y_{d,s,t} \\ +\tau^{Reg}_{d,s,t}\left(1+\tau^{VATM}_{d,s,t}\right)\left(1+\tau^{DutyM}_{d,s,t} - s^{M}_{d,s,t}\right)\left(1+\tau^{Cus}_{d,s,t}\right) P^{M}_{s,t} M_{d,s,t} \end{array}\right]$$

where the exogenous implicit car duty rates $\tau^{Reg}_{d,s,t}$ are 0 for all but three demand components: Private consumption of cars, investments in equipment, and government consumption.

Production taxes are taxes on the production input (labor, capital and materials) of firms. They are sector specific and consist of taxes on land (grundskyld), $T^{Land}_{s,t}$, weight charges on cars, $T^{FirmWeight}_{s,t}$, payroll taxes, $T^{Payroll}_{s,t}$, taxes related to firm's contribution to workers education, $T^{FirmEdu}_{s,t}$, and a small sum of other production taxes, $T^{ProductionRest}_{s,t}$:

$$T^{Production}_{t} = \sum_{s}\left(T^{Land}_{s,t} + T^{FirmWeight}_{s,t} + T^{Payroll}_{s,t} + T^{FirmEdu}_{s,t} + T^{ProductionRest}_{s,t}\right)$$

The tax on land is modeled as:

$$T^{Land}_{s,t} = \tau^{Land}_{s,t} P^{I}_{iB,s,t} K_{iB,s,t-1}$$

With an exogenous tax rate, we assume that the tax base for land value follows the replacement value of structures capital in the relevant sector, $P^{I}_{iB,s,t} K_{iB,s,t-1}$.

Both the payroll tax and the education contributions are proportional to payrolls:[88]

$$T_{s,t}^{Payroll} = \tau_{s,t}^{Payroll} \cdot Payroll_{s,t}$$

$$T_{s,t}^{FirmEdu} = \tau_{s,t}^{FirmEdu} \cdot Payroll_{s,t}$$

The residual production taxes follow gross value added in each sector:

$$T_{s,t}^{ProductionRest} = \tau_{s,t}^{ProductionRest} GVA_{s,t}$$

Indirect taxes to the EU is not exactly equal to customs so a correction factor is added:[89]

$$T_t^{EU} = f_t^{TCus} \cdot T_t^{Cus}$$

10.1.4 Other government revenues

Other revenues are as follows:

$$\begin{aligned} T_t^{Other} = \quad & T_t^{Bequest} + T_t^{\delta} + Cont_t + T_t^{ForeignCap} \\ & + T_t^{ForeignEU} + T_t^{ForeignRest} + T_t^{Hh} \\ & + T_t^{Firms} + T_t^{Church} + T_t^{LandRent} + T_t^{GovFirms} \end{aligned}$$

Bequest taxes (arveafgift) follow the bequest amount, $T_t^{Bequest} = \tau_t^{Bequest} \cdot Beq_t$, where $\tau_t^{Bequest}$ is the implicit tax rate and Beq_t is the sum of bequests described in the household chapter.

Revenues from the depreciation of government capital T_t^{δ} are discussed in the chapter on government production and consist of depreciation allowances paid by the government to itself. They are included here as revenue, while on the expenditure side they are a part of government consumption. On the public production side depreciation is counted as a cost. The public sector gets the money back here, however so the actual capital expense is the investment.

Contributions to social programs (Bidrag til sociale ordninger) $Cont_t$, sums a list of different specific payments to the state:

$$\begin{aligned} Cont_t = \quad & Cont_t^{Unemp} + Cont_t^{CivilServants} + Cont_t^{EarlyRet} \\ & + Cont_t^{FreeRest} + Cont_t^{Mandatory} \end{aligned}$$

All these contributions follow the labor force through a relation of the form:[90]

---

[88] The payrolls are given by: $Payroll_{s,t} = \left(1 - \lambda_{s,t}^{SelfEmp}\right) w_t \rho_{s,t} h_{s,t} n_{s,t}$, where $\lambda_{s,t}^{SelfEmp}$ is the exogenous share of self-employed in sector s. For notation on other parameters refer to labor market chapter.

[89] We have almost exactly $T_t^{Cus} = T_t^{EU}$.

[90] The labor force in MAKRO is a sum of people in socio-economic groups in the labor force $\sum_{soc \in \text{Labor Force}} n_{soc,t}$ plus foreign worker. The number of people in socio-economic groups depends mechanically on employment and population as described under transfers in the expenditure section of this chapter.

$$Cont_t^X = \mu_t^X \cdot n_t^{LabForce}$$

with contribution rate μ_t^X.[91] The index X contains contributions to early retirement, other voluntary contributions (øvrige frivillige bidrag), mandatory contributions (obligatoriske bidrag), and contributions to civil servant pensions (bidrag til tjenestemandspension).

Transfers to the government are divided into foreign capital transfers, $T_t^{ForeignCap}$, non-capital foreign transfers from the EU, $T_t^{ForeignEU}$, other foreign transfers, $T_t^{ForeignRest}$, transfers from households, T_t^{Hh}, transfers from firms, T_t^{Firms}, and transfers from publicly owned firms, $T_t^{GovFirms}$.

All these transfers are all calibrated to match their respective shares of GDP:

$$T_t^X = \alpha_t^X GDP_t$$

with $X \in \{ForeignCap, ForeignEU, ForeignRest, Hh, Firms, GovFirms\}$.

Tax revenue from the church tax follows the same tax base as municipal taxation and is (at a personal level) given by:

$$T_{a,t}^{Church} = \tau_t^{Church} r_t^{TChurch} \left(y_{a,t}^{Taxable} - y_{a,t}^{PA} \right)$$

where $r_t^{TChurch}$ are the share of tax-payers that pay church-taxes.

The land rent, $T_t^{LandRent}$, depends on the gross value added in the extraction sector, and is given by:

$$T_t^{LandRent} = \tau_t^{LandRent} \cdot GVA_{extraction,t}$$

10.2 Expenditures

Total government expenditures, Exp_t^G, are the sum of expenditures on government consumption, $P_t^G G_t$, income transfers, TR_t, value of investments, $\sum_k P_{k,\text{Pub},t}^I I_{k,\text{Pub},t}$, subsidies, Sub_t, and other expenses, Exp_t^{Other}:

$$\text{Expenditures} \equiv Exp_t^G = P_t^G G_t + TR_t + \sum_k P_{k,\text{Pub},t}^I I_{k,\text{Pub},t} + Sub_t + Exp_t^{Other}$$

The largest expense categories are government consumption and income transfers. In 2021 these two posts were respectively 49 and 32 percent of government expenditures.

Government consumption, G_t, and investments, $I_{k,\text{Pub},t}$, are exogenous. The prices of these are, as usual for demand prices, given by the output prices from the sectors delivering inputs plus indirect taxes.[92]

[91] This rate follows the regulation rate of public transfers (sats-regulering) as explained under public expenditures.
[92] Expenditures on government consumption and investments depend on the endogenous output prices, where

Further details of government investments and capital stocks are given in the chapter on public production.

Income transfers, subsidies and other expenses are described in the rest of this section.

10.2.1 Income transfers

Government income transfers are the sum of many different types[93]:

$$TR_t = \sum_j TR_{j,t}$$

Every type of income transfer j is determined as a rate per person in million DKK times a number of recipients in thousand persons:[94]

$$TR_{j,t} = Rate_{j,t} \cdot n_{j,t}$$

Rates follow the wage index, s_t^{reg}, (satsregulering) [95]:

$$Rate_{j,t} = \mu_{j,t}^{Rate} s_t^{reg}$$

where $\mu_{j,t}^{Rate}$ is a calibrated parameter.

The wage index is based on the average wage per worker with a two year lag:

$$s_t^{reg} = s_{t-1}^{reg} \frac{W_{t-2}/n_{t-2}^e}{W_{t-3}/n_{t-3}^e}$$

The number of recipients is based on an exogenous mapping $\frac{\partial n_j}{\partial n_{soc}}$ from socio-economic groups[96] contained in the demographic projection (befolkningsregnskab) to the income transfer type:

$$n_{j,t} = \sum_{soc} \underbrace{\frac{\partial n_j}{\partial n_{soc}}}_{\text{exogenous}} \cdot n_{soc,t}$$

The mapping $\frac{\partial n_j}{\partial n_{soc}}$ is a $j \times soc$ matrix. In most cases this matrix only has diagonal elements - i.e. one socio-economic group receives one type of transfer. In several cases, however, more than one socio-economic group receives the same transfer type, for example employed and unemployed students both receive student benefits. Also, the base for some transfers is all

the public sector weighs the most. The public sector's output price follows its production costs, which are mainly wages. So expenditures on government consumption and investments follow the wage level relatively close even though not one to one.

93 In the current model version there are 36 different income transfers: 12 types of unemployment compensation, 7 types of pension and 17 other including student aid, support for housing, children etc.

94 It is necessary to include an adjustment term in order to calibrate the model as in some years transfers have been paid even though the base is zero. This is probably due to corrections in transfers paid from the year before. The numbers are, however, very small and in projections this adjustment term is set to zero and not used.

95 The only exception is the j = Green Check, which just consists of a constant value $Rate_{\text{Green Check},t} = \mu^{Rate}_{\text{Green Check},t}$.

96 In the current model version there are 52 different socio-economic groups. The employed are divided into 19 groups, the unemployed into 11 groups, leaving 22 groups for people outside the labor market.

people of age 18 and above. In a few cases the socio-economic groups are divided between two transfer groups where they are not the only recipients. This makes it necessary to have coefficients between one and zero in a few cells.

The number of persons in the different socio-economic groups changes with employment. When employment increases by 1.000 persons groups, that make up employment increase by 1.000 persons, and the groups which make up non-employment decrease by the same 1.000 persons. The specific allocation follows the deviation from structural employment:

$$n_{soc,t} = n^*_{soc,t} + \underbrace{\frac{\partial n_{soc,t}}{\partial n^e_t}}_{\text{exogenous}} \left(n^e_t - n^{e*}_t\right)$$

where n^e_t is total employment (excluding non-resident workers) and n^{e*}_t is the equivalent structural employment. The factor $\frac{\partial n_{soc,t}}{\partial n^e_t}$ is the marginal effect from deviations of employment relative to its structural level on the composition of the different population groups (socio-economic, index soc).[97] The object $n^*_{soc,t}$ is the structural number of persons in the socio-economic group soc.

10.2.2 Age distribution of income transfers

The previous section covered income transfers by the type of income transfer, j. The total of these income transfers to households is $\sum_j TR^G_{j,t}$. We now look at how these transfers are distributed among households of different ages.

To keep the model tractable, we do not model each type of transfer by age endogenously. Instead, we model only the total transfers received by age, $y^G_{a,t}$. Total transfers by age must of course match total transfers by transfer type; that is

$$\sum_a y^G_{a,t} N_{a,t} = \sum_j TR^G_{j,t}$$

In the baseline forecast, we do however, utilize a full demographic forecast by age and socio-economic group, $n_{soc,a,t}$. In the baseline, we generally assume that the rate received for a particular income transfer is independent of age, but that the share of recipients varies greatly by age. We can denote the full transfer matrix each period by age and type of transfer as $y^G_{j,a,t}$.

The total income transfer received by an individual is

[97] The marginal effects are measured empirically by the Ministry of Finance using correlations with the employment gap. This is described in more detail in the working paper: "Fordeling af ændringer i beskæftigelsen på socioøkonomiske grupper" published on www.fm.dk.

$$y^G_{a,t} = \sum_j y^G_{j,a,t}$$

$$= \sum_j Rate_{j,t} \cdot \frac{n_{j,a,t}}{N_{a,t}}$$

$$= \sum_j \left(Rate_{j,t} \sum_{soc} \underbrace{\frac{\partial n_j}{\partial n_{soc}}}_{\text{exogenous}} \cdot \frac{n_{soc,a,t}}{N_{a,t}} \right)$$

For marginal shocks to the model, we model the row and column totals of the $y^G_{j,a,t}$ matrix, but leave the age×transfer cells unspecified. As already stated, we do this in order to keep the model tractable.

Specifically, we model $y^G_{a,t}$ as

$$y^G_{a,t} = \frac{\partial y^G_t}{\partial n^e_t} \cdot \frac{n^e_{a,t}}{N_{a,t}} + \frac{\partial y^G_t}{\partial N_t} \mu^{yG}_{a,t} + y^{GOther}_{a,t}$$

where $\frac{\partial y^G_t}{\partial n^e_t}$ and $\frac{\partial y^G_t}{\partial N_t}$ are endogenous and depend on $Rate_{j,t}$ and $\mu^{yG}_{a,t}$ is a calibrated parameter. The term $y^{GOther}_{a,t}$ contains age-specific transfers that are not included in the mapping from socio-economic groups ($j \in \{$Child-base, Adult-base$\}$ described above).

The endogenous derivatives, $\frac{\partial y^G_t}{\partial n^e_t}$ and $\frac{\partial y^G_t}{\partial N_t}$, are given by:

$$\frac{\partial y^G_t}{\partial n^e_t} = \sum_j Rate_{j,t} \frac{\partial n_{j,t}}{\partial n^e_t}$$
$$= \sum_j \left\{ Rate_{j,t} \sum_{soc} \underbrace{\frac{\partial n_j}{\partial n_{soc}}}_{\text{exogenous}} \underbrace{\frac{\partial n_{soc,t}}{\partial n^e_t}}_{\text{exogenous}} \right\}$$

and

$$\frac{\partial y^G_t}{\partial N_t} = \sum_j Rate_{j,t} \frac{\partial n_{j,t}}{\partial N_t}$$
$$= \sum_j \left\{ Rate_{j,t} \sum_{soc} \underbrace{\frac{\partial n_j}{\partial n_{soc}}}_{\text{exogenous}} \underbrace{\frac{\partial n_{soc,t}}{\partial N_t}}_{\text{exogenous}} \right\}$$

where the derivatives $\frac{\partial n_{soc,t}}{\partial n^e_t}$ and $\frac{\partial n_{soc,t}}{\partial N_t}$ are exogenous when we shock the model. We covered the estimated relation $\frac{\partial n_{soc,t}}{\partial n^e_t}$ in the previous section. We calibrate the residual $\frac{\partial n_{soc,t}}{\partial N_t}$ to match an exogenous baseline forecast of structural socio-economic groups, such that

$$n^*_{soc,t} = \underbrace{\frac{\partial n_{soc,t}}{\partial n^e_t}}_{\text{exogenous}} n^{e*}_t + \underbrace{\frac{\partial n_{soc,t}}{\partial N_t}}_{\text{exogenous}} N_t$$

When analyzing effects of structural reforms which directly affect the number of transfer recipients (for example reforms regarding pension and education), users can directly specify the expected change in structural socio-economic groups $n^*_{soc,t} = \sum_a n^*_{a,soc,t}$ and calculate $\frac{\partial n_{soc,t}}{\partial N_t}$ residually to match the change.

Not all government income transfers are subject to income taxes. The set of non-taxed transfers is a subset of the index j used above - let us call it j^-:

$$TR^{NonTax}_t = \sum_{j^-} TR_{j^-,t}$$

This is distributed by age according to a exogenous distribution, $\alpha^{GNonTax}_{a,t}$:

$$y^{GNonTax}_{a,t} = \alpha^{GNonTax}_{a,t} TR^{NonTax}_t$$

where $\sum_a \alpha^{GNonTax}_{a,t} N_{a,t} = 1$ such that $\sum_a y^{GNonTax}_{a,t} N_{a,t} = TR^{NonTax}_t$ and taxable transfers are given by

$$y^{GTax}_{a,t} = y^G_{a,t} - y^{GNonTax}_{a,t}$$

10.2.3 Subsidies

Government subsidies are subsidies for products and production minus subsidies financed by the EU:

$$Sub_t = Sub^{Product}_t + Sub^{Production}_t - Sub^{EU}_t$$

Product subsidies are added simultaneously with excise duties after customs. The expense is modeled as:

$$Sub^{Product}_t = \sum_d \sum_s \left[s^Y_{d,s,t} P^Y_{s,t} Y_{d,s,t} + s^M_{d,s,t} \left(1 + \tau^{Cus}_{d,s,t}\right) P^M_{s,t} M_{d,s,t} \right]$$

For notation, refer to subsection 10.1.3 on indirect taxes above.

Production subsidies are sector specific and related to firm production:

$$Sub^{Production}_t = \sum_s \varsigma^{Wage}_{s,t} Payroll_{s,t} + \sum_s \varsigma^{ProductionRest}_{s,t} GVA_{s,t}$$

where $\varsigma_{s,t}^{Wage}$ and $\varsigma_{s,t}^{ProductionRest}$ are exogenous effective subsidy rates and $GVA_{s,t}$ is gross value added by sector s.

Subsidies financed by the EU are exogenous in MAKRO.

10.2.4 Other expenses

Other expenses consist of government land purchases and non-income transfers to abroad, households and domestic firms:

$$Exp_t^{Other} = Exp_t^{Land} + Exp_t^{Foreign} + Exp_t^{Hh} + Exp_t^{Firms}$$

The non-income transfers to abroad are divided into sub-categories:

$$\begin{aligned} Exp_t^{Foreign} \quad = \quad & Exp_t^{ForignCap} + Exp_t^{ForeignVAT} + Exp_t^{ForeignGNI} + Exp_t^{ForeignEU} \\ & + Exp_t^{ForignFO} + Exp_t^{ForeignGL} + Exp_t^{ForeignAid} \end{aligned}$$

where $Exp_t^{ForignCap}$ are capital transfers, $Exp_t^{ForeignVAT}$ are VAT contributions to the EU, $Exp_t^{ForeignGNI}$ are gross net income contributions to the EU, $Exp_t^{ForeignEU}$ are other transfers to the EU, $Exp_t^{ForeignFO}$ are transfers to the Faroe Islands, $Exp_t^{ForeignGL}$ are transfers to Greenland and $Exp_t^{ForeignAid}$ are transfers given as foreign aid.

The non-income transfers to the households are also divided into sub-categories:

$$Exp_t^{Hh} = Exp_t^{HhCap} + Exp_t^{HhNPISH} + Exp_t^{HhIndex} + Exp_t^{HhRest}$$

where Exp_t^{HhCap} are capital transfers, $Exp_t^{HhNPISH}$ are transfers to NPISH (Non-Profit Organizations Serving Households), $Exp_t^{HhIndex}$ are transfers to households via index supplements to pension schemes, and Exp_t^{HhRest} are other residual transfers to households.

All the non-income transfers mentioned in this sub-section are exogenous in MAKRO.

10.3 Net interest income and government net wealth

Net interest income consists of earned interest income from government assets, $A_{i,t-1}^{G}$, minus paid interests on government liabilities, $L_{j,t-1}^{G}$:

$$\text{Net Interests} \equiv NetInt_t^G = \sum_i A_{i,t-1}^G r_{i,t}^{Interest} - \sum_j r_{j,t}^{Interest} L_{j,t-1}^G$$

where $r_{i,t}^{Interest}$ is interest for bonds and deposits and dividends for equity.

Government assets are exogenous and consist of bonds, deposits, and equity (almost exclusively domestic). Government liabilities consist only of bonds (divided between real estate bonds and other bonds) and are endogenously given by the government net wealth:

$$W_t^G = A_{t-1}^G - L_{t-1}^G$$

The net wealth is given by the government budget:[98]

$$W_t^G = W_{t-1}^G + \underbrace{T_t - Exp_t^G + NetInt_t^G}_{\text{Budget}} + \underbrace{\sum_i A_{i,t-1}^G r_{i,t}^{Reval} - \sum_j L_{j,t-1}^G r_{j,t}^{Reval}}_{\text{Revaluations}}$$

where $r_{i,t}^{Reval}$ is revaluation of rate of assets and liabilities where $r_{i,t}^{Return} = r_{i,t}^{Interest} + r_{i,t}^{Reval}$. It should be noted that since government assets are exogenous and only government liabilities of bonds are endogenous, the marginal government interest rate is that of bonds.

10.4 The fiscal sustainability indicator

The fiscal sustainability indicator, in the model called HBI (holdbarhedsindikator), is equal to the net present value of all future government revenues minus expenditures relative to the net present value of future GDP.

$$HBI_t = \frac{S_t + O_t - R_t^{Marginal} + (1 + r_t) \times GovNetAssets_{t-1}}{Y_t}$$

In this formula, $GovNetAssets_{t-1}$ is the value of government net financial assets. S_t, O_t, $R_t^{Marginal}$, and Y_t represent the respective net present values of the primary balance, public revaluations, interest margin, and GDP for all future periods. These values are determined by the following forward-looking difference equations:

$$S_t = PrimBalance_t + \frac{S_{t+1}}{1 + r_t} \left(= \sum_{s=t}^{\infty} \Gamma_{s-1} \times PrimBalance_s \right)$$

$$O_t = PubReval_t + \frac{O_{t+1}}{1 + r_t} \left(= \sum_{s=t}^{\infty} \Gamma_{s-1} \times PubReval_s \right)$$

$$R_t^{Marginal} = IntMarg_t + \frac{R_{t+1}^{Marginal}}{1 + r_t} \left(= \sum_{s=t}^{\infty} \Gamma_{s-1} \times IntMarg_s \right)$$

$$Y_t = GDP_t + \frac{Y_{t+1}}{1 + r_t} \left(= \sum_{s=t}^{\infty} \Gamma_{s-1} \times GDP_s \right)$$

Where $PrimBalance_t$ is the primary balance, $PubReval_t$ is the public revaluation, $IntMarg_t$ is the interest margin, and GDP_t is the gross domestic product. The discount factor Γ_t is given by the backward looking difference equation:

[98] The equation is the code also includes corrections for net government wealth defined in national accounts vs. statistics on public finances.

$$\Gamma_t = \frac{1}{1+r_t}\Gamma_{t-1}$$

In the terminal period (currently set to 2130) stationary state is assumed such that:

$$S_T = PrimBalance_T + \frac{S_T}{1+r_T}$$

$$O_T = PubReval_T + \frac{O_T}{1+r_T}$$

$$R_T^{Marginal} = IntMarg_T + \frac{R_T^{Marginal}}{1+r_T}$$

$$Y_T = GDP_T + \frac{Y_T}{1+r_T}$$

11 References

- Anderson, James E., 1979. “A Theoretical Foundation for the Gravity Equation,” American Economic Review, 69(1), 106–116.
- Anderson, James, E., and Eric van Wincoop. 2003. "Gravity with Gravitas: A Solution to the Border Puzzle ." American Economic Review, 93 (1): 170-192.
- Ampudia, M., Cooper, R., Le Blanc, J., and Zhu, G. (2019). MPC Heterogeneity in Europe: Sources and Policy Implications.
- Armington, P. S., 1969: “A Theory of Demand for Products Distinguished by Place of Production,” International Monetary Fund Staff Papers, 16, 159–178.
- Attanasio, O., Peter Levell, Hamish Low, Virginia Sanchez-Marcos (2018). Aggregating Elasticities: Intensive and Extensive Margins of Women’s Labour Supply. June 4, 2018.
- Bernanke, B. and M. Gertler (1989): Agency Costs, Net Worth, and Business Fluctuations," American Economic Review 79, 14-31.
- Blanchard, O., and Fischer, S. (1989), Lectures on Macroeconomics. MIT Press.
- Blanchard, O. (2018). On the future of macroeconomic models. Oxford Review of Economic Policy, Volume 34, Issue 1-2, Spring-Summer 2018, Pages 43–54.
- Blanchard, O. (2017), “Should we get rid of the natural rate hypothesis?”, NBER Working Paper 24057, November 2017.
- Blanchard, O., and Galí, J., 2008. "Labor Markets and Monetary Policy: A New-Keynesian Model with Unemployment," NBER Working Papers 13897, National Bureau of Economic Research, Inc.
- Boldrin, Michelle, Lawrence J. Christiano, and Jonas Fisher “Habit Persistence, Asset Returns, and the Business Cycle,” American Economic Review, 91: 149—166, 2001. 23
- Boserup, S., Kpoczuk, W., Kreiner, C. (2016), "The Role of Bequests in Shaping Wealth Inequality: Evidence from Danish Wealth Records," American Economic Review, American Economic Association, vol. 106(5), pages 656-661, May.
- Browning, M., & Ejrnæs, M. (2009). Consumption and Children. Review of Economics and Statistics, 91(1), 93-111.
- Campbell, John Y., and Mankiw, N. Gregory, (1989). Consumption, Income and Interest Rates: Reinterpreting the Time Series Evidence. NBER Macroeconomics Annual 1989, Volume 4, (p. 185 - 246).
- Carroll, Christopher. (2001). Death to the Log-Linearized Consumption Euler Equation! (And Very Poor Health to the Second-Order Approximation). Advances in Macroeconomics. 1. 1003-1003. 10.2202/1534-6013.1003.

- Christiano, L., Eichenbaum, M., and Trabant, M., (2016), Unemployment and Business Cycles, Econometrica, July 2016, 84(4), pp. 1523-1569.
- Cooper, R., and Adda, J. (2000), "Balladurette and Juppette: A Discrete Analysis of Scrapping Subsidies", Journal of Political Economy Volume 108, Number 4 | August 2000.
- Coibion, O., and Gorodnichenko, Y. (2013) "Is the Phillips Curve alive and well after all?", NBER Working Paper 19598, October 2013.
- den Haan, W. J., Ramey, G., & Watson, J. (2000) "Job Destruction and Propagation of Shocks", American Economic Review 90(3), 482-498.
- Dennis, R. (2009) Consumption habits in a new keynesian business cycle model. Journal of Money, Credit and Banking.
- Drechsel, T. (2021). Earnings-Based Borrowing Constraints and Macroeconomic Fluctuations. University of Maryland. (2019 LSE Job Market paper)
- Eaton, J., & Kortum, S. (1999). International Technology Diffusion: Theory and Measurement. International Economic Review, 40(3), 537-570.
- Eaton, J., & Kortum, S. (2002). Technology, Geography, and Trade. Econometrica, 70(5), 1741-1779.
- Galí, J., and Gertler, M. (1999), Inflation Dynamics: A Structural Econometric Analysis. Journal of Monetary Economics Volume 44, Issue 2, October 1999, Pages 195-222.
- Galí, J., Smets, F., and Wouters, R., (2012), "Unemployment in an Estimated New Keynesian Model," NBER Macroeconomics Annual 26 (2012): 329-360.
- Gomes, J. F. (2001). Financing Investment. American Economic Review 91, 1263-1285.
- Gordon, R. (2013) "The Phillips Curve is alive and well", NBER Working Paper 19390, August 2013.
- Gourinchas, P., and Parker, J. (2002). Consumption Over the Life Cycle. Econometrica. 70. 47-89.
- Kaplan, G., G., Mitman, K., and Violante, G. (2017), The Housing Boom and Bust: Model meets Evidence. NBER Working Paper 23694, August 2017.
- Kaplan, G., Violante, G. L., and Weidner, J., (2014). The Wealthy hand-to-mouth. NBER Working Paper 20073, April 2014.
- King, Plosser and Rebelo (1988), "Production, Growth and Business Cycles", Journal of Monetary Economics 21, 195-232.
- Kiyotaki, N., and Moore, J., (1997). Credit Cycles. Journal of Political Economy Vol. 105, No. 2, pp. 211-248.
- Kocherlakota, N. (2010). Modern Macroeconomic Models as Tools for Economic Policy. Federal Reserve Bank of Minneapolis. Annual Report Essay May 4, 2010. Narayana Kocherlakota, President, Minneapolis Federal Reserve Bank.

- Kravik, Erling Motzfeldt og Yasin Mimir (2019). “Navigating with NEMO”.
- Krugman, Paul R., 1979. "Increasing returns, monopolistic competition, and international trade," Journal of International Economics, Elsevier, vol. 9(4), pages 469-479, November.
- Krugman, P., 1980. “Scale Economies, Product Differentiation, and the Pattern of Trade,” American Economic Review, 70(5), 950–959.
- Kronborg, A., Poulsen, K., & Kastrup, C. (2020). “Estimering af udenrigshandelselasticiteter i MAKRO”. Working paper, DREAM.
- Li, W., Liu, H., Yang, F., and Yao, R. (2016), “Housing over Time and over the Life Cycle: A Structural Estimation”, International Economic Review, Volume 57, Issue 4, Pages 1237-1260.
- Lian, C. and Yueran Ma (2020), Anatomy of Corporate Borrowing Constraints. The Quarterly Journal of Economics, vol. 136, issue 1, 229-291
- Ljungqvist, L., and Sargent, T. (2017). "The Fundamental Surplus," American Economic Review, American Economic Association, vol. 107(9), pages 2630-2665, September.
- Mankiw, G. (1982), “Hall’s Consumption Hypothesis and Durable Goods”, Journal of Monetary Economics 10, 417-425.
- Melitz, M., 2003, The impact of trade on intra-industry reallocations and aggregate industry productivity. Econometrica 71, Number 6, pp. 1695-1725.
- Mortensen, D., and Pissarides, C. (1994). Job Creation and Job Destruction in the Theory of Unemployment. The Review of Economic Studies, 61(3), 397-415.
- Mortensen, Anne Ulstrup (2015) “The Keynesian Multiplier in a CGE Model”, Master Thesis, Institute of Economics, University of Copenhagen.
- Mortensen, M., Sisay, D., and Kristensen, T., 2014. “Export market index: forecast for latest historical period”, Working Paper, Danmarks Statistik Modelgruppen.
- Pedersen, J. (2016). An Estimated DSGE Model for Denmark with Housing, Banking, and Financial Frictions, Danmarks Nationalbank Working Paper N. 108, 2016.
- Petrongolo, B., and Pissarides, C., (2001), Looking into the Black Box: A Survey of the Matching Function, Journal of Economic Literature Vol. 39, No. 2, June 2001, (pp. 390-431)
- Rebelo, S. (2005), “Real Business Cycle Models: Past, Present, and Future”, The Scandinavian Journal of Economics 107(2), 217-238.
- Rogerson, R. (1988). Indivisible Labor, Lotteries and Equilibrium. Journal of Monetary Economics 21, pp 3-16.
- Rotemberg, J., (1982). “Monopolistic Price Adjustment and Aggregate Output”. Review of Economic Studies 49(4), 517–531.
- Sisay, D., 2013. “Export Market and Market Price Indices for ADAM”, Working Paper, Danmarks Statistik Modelgruppen.

- Straathof, Bas., 2008. "Gravity with gravitas: comment," CPB Discussion Paper 111.rdf, CPB Netherlands Bureau for Economic Policy Analysis.
- Temere, Dawit S., and Tony M. Kristensen. 2016. "Supply factors in trade determination." Danmarks Statistik Modelgruppen, Working Paper DSI080816, August 2016.
- Temere, Dawit S., and Tony M. Kristensen. 2016. "Incorporating supply effects in ADAM's export equations." Danmarks Statistik Modelgruppen, Working Paper DSI111116, November 2016.
- Vetlov, I., Hlédik, T., Jonsson, M., Kucsera, H. and Pisani, M., Potential Output in DSGE Models (May 26, 2011). ECB Working Paper No. 1351
- Yang, F. (2009), "Consumption over the life cycle: How different is housing?", Review of Economic Dynamics, Volume 12, Issue 3, July 2009, Pages 423-443

DREAM

Danish Research institute for
Economic Analysis and Modelling

DREAM
Landgreven 4

1301 Copenhagen K
Denmark
Phone: +45 25 26 27 60
info@dreamgruppen.dk
www.dreamgruppen.dk